Counting Statistics of an Adiabatic Pump

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We use the Schwinger-Keldysh formalism to derive the charge counting statistics of an adiabatic pump based on an open quantum dot. The distribution function of the transmitted charge in terms of the time-dependent S matrix is obtained. It is applied to a few simple examples of the pumping cycles. By a *chiral* gauge transformation the problem is mapped onto a problem of pumping by voltage pulses. The role of the chiral anomaly arising in this mapping is emphasized. Conditions for the ideal noiseless quantized pump are discussed.

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Adiabatic charge pumping has attracted considerable theoretical and experimental interest. It occurs when the Hamiltonian of the system is changed periodically with time. At the end of the pumping cycle a finite charge may be transmitted through the system. The idea is originally due to Thouless [1], who showed that in certain one-dimensional systems the transmitted charge is quantized in the adiabatic limit.

The majority of research efforts have focused on adiabatic pumping through mesoscopic devices [2–8]. Such mesoscopic pumps are based on one or several quantum dots connected to leads. Motivated by efforts to build a standard of electric current most investigations concentrated on closed devices where the Coulomb blockade effects play an essential role and can lead to quantization of the transmitted charge [6–8].

Recently adiabatic pumping through open devices, where the Coulomb blockade effects can be neglected, also became a subject of both experimental and theoretical investigations [3–5] which focused mainly on the average charge transmitted during a pumping cycle and its mesoscopic fluctuations. Thermal and quantum fluctuations give rise to equilibrium and shot noise in the transmitted charge. The fluctuations in the transmitted charge can be measured and require a better understanding in view of potential applications of adiabatic pumps as standards of electric current.

In this Letter, we apply the Schwinger-Keldysh formalism to the problem of charge counting statistics of an adiabatic pump based on an open quantum dot [3,5].

We have greatly benefited from the extensive work on the charge counting statistics by Levitov *et al.* [9–11].

We consider an adiabatic pump connected to the left and right leads by multichannel quantum point contacts, as in Ref. [5], each having n transverse channels [12]. Since the conductances of the contacts are greater than the conductance quantum e^2/h one can neglect the Coulomb blockade effects. Furthermore, the level broadening due to the electron-electron interaction in the dot becomes comparable to the mean level spacing only for electrons with energy greater than the Thouless energy E_c in the dot [13]. Since the dwell time in the device is shorter than the inverse mean level spacing the inelastic level broadening can be neglected if the temperature and the frequency of pumping are smaller than E_c. The pump can therefore be described by a $2n \times 2n$ unitary single-particle scattering matrix $S(\tau, \epsilon)$, where ϵ is the electron energy, and the S matrix is a periodic function of time τ with period τ_0; $S(\tau + \tau_0, \epsilon) = S(\tau, \epsilon)$.

In the adiabatic limit, when τ_0 exceeds the flight time across the dot $\tau_W = \hbar \Im \, \mathrm{Tr}\{S^\dagger \partial S/\partial \epsilon\}$ one can neglect the energy dependence of the scattering matrix. We therefore omit the energy argument of the S matrix and write it in the $n \times n$ block form

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}. \tag{1}$$

Here r (r') and t (t') are left (right) reflection and transmission matrices, correspondingly.

The transmission and reflection matrices in Eq. (1) can be simultaneously diagonalized by the *block-diagonal* unitary matrices $U(\tau)$ and $V(\tau)$ (see, for example, Ref. [14]),

$$S(\tau) = U(\tau)\tilde{S}(\tau)V^\dagger(\tau). \tag{2}$$

Here $\tilde{S}(\tau)$ is a matrix of the form Eq. (1) with real diagonal reflection and transmission blocks. The ambiguity in the definition of matrices $U(\tau)$ and $V(\tau)$ does not affect our results.

In each cycle a number of electrons Q may pass through the scattering region in the direction which depends on the detailed form of $S(\tau)$. The average charge transmitted in N pumping cycles was recently given by Brouwer [3]:

$$\langle Q \rangle = \frac{1}{2i} \int_0^{N\tau_0} \frac{d\tau}{2\pi} \mathrm{Tr}\left\{ \frac{\partial S}{\partial \tau} S^\dagger \sigma_3 \right\}, \tag{3}$$

where $\sigma_3 = \mathrm{diag}\{1, -1\}$ is a Pauli matrix with the $n \times n$ block structure and the charge Q is measured in units of the electron charge. We have used the representation where the transmitted charge is written as half the sum of that through the left and through the right leads [15]. Both quantum and thermal noise lead to fluctuations in the transmitted charge Q. As a result the charge transmitted in one cycle can be described by a certain probability distribution.

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Here we calculate the probability $P_N(Q)$ to transmit the charge Q upon completion of N pumping cycles. It is convenient to formulate the results in terms of the generating function $F_N(\lambda)$ of the moments of the transmitted charge defined as [9]

$$F_N(\lambda) = \langle e^{iQ\lambda} \rangle = \int dQ\, P_N(Q) e^{iQ\lambda}. \tag{4}$$

For the S matrix of the form Eq. (2) we obtain

$$F_N(\lambda) = e^{i\hat{N}\lambda} \det[1 + \tilde{S}_\lambda(\tau)\tilde{n}(\tau,\tau') \times [\tilde{S}^\dagger_{-\lambda}(\tau') - \tilde{S}^\dagger_{\lambda}(\tau')]], \tag{5}$$

where

$$\tilde{S}_\lambda(\tau) \equiv \exp\{-i\sigma_3\lambda/4\}\tilde{S}(\tau)\exp\{i\sigma_3\lambda/4\}. \tag{6}$$

The matrix $\tilde{n}(\tau,\tau')$ is defined as

$$\tilde{n}(\tau,\tau') = V^\dagger(\tau)\hat{n}(\tau - \tau')V(\tau'). \tag{7}$$

Here the diagonal matrix $\hat{n}(\tau - \tau')$ is the time Fourier transform of $\hat{n}(\epsilon) = \mathrm{diag}\{n_L(\epsilon), n_R(\epsilon)\}$, where $n_{L(R)}(\epsilon)$ is the energy distribution function of the left (right) lead. The operator in the determinant in Eq. (5) should be understood as an operator in the time space as well as a matrix in the space of channels. Finally, the *integer* number $\hat{N}$ defined as

$$\hat{N} = \frac{1}{2i}\int_0^{N\tau_0} \frac{d\tau}{2\pi} \mathrm{Tr}\left\{U^\dagger \frac{\partial U}{\partial\tau}\sigma_3 - V^\dagger\frac{\partial V}{\partial \tau}\sigma_3\right\}, \tag{8}$$

is the contribution to the generating function arising from the chiral anomaly (see below).

For the case of time-independent reflection matrix the charge counting statistics were obtained in Ref. [10]. Equation (5) differs from the result of Ref. [10] by the chiral anomaly term, Eq. (8), and generalizes the result of Ref. [10] to general pumping cycles.

Expanding $\ln F_N(\lambda)$ to the first power of $i\lambda$ one finds, for the average transmitted charge, $\langle Q \rangle \equiv \sum_Q QP_N(Q)$,

$$\langle Q \rangle = \int_0^{N\tau_0} \frac{d\tau}{4\pi} \mathrm{Tr}\{\tilde{S}(\tau)\tilde{n}(\tau,\tau')[\sigma_3, \tilde{S}^\dagger(\tau')]\}|_{\tau'\to\tau} + \hat{N}. \tag{9}$$

In the absence of an external voltage $\tilde{n}(\tau,\tau') \to i/[2\pi(\tau - \tau' + i\eta)] + i/(2\pi)V^\dagger(\tau)\partial V(\tau)/\partial\tau$ for $\tau' \to \tau$. Expanding $\tilde{S}^\dagger(\tau')$ to the first power in $\tau - \tau'$ and using Eq. (8) one obtains Brouwer's result [3], Eq. (3).

The most readily measurable characteristic of the pumped charge noise is its variance, $\langle\langle Q^2 \rangle\rangle$. Expanding $\ln F_N(\lambda)$, Eq. (5), to the second order in $i\lambda$ we obtain [9]

$$\langle\langle Q^2 \rangle\rangle = \int\int_0^{N\tau_0} \frac{d\tau\, d\tau'}{(4N\tau_0)^2} \frac{\mathrm{Tr}\{1 - (S^\dagger\sigma_3 S)_\tau (S^\dagger\sigma_3 S)_{\tau'}\}}{\sin^2[\pi(\tau - \tau')/(N\tau_0)]}. \tag{10}$$

We note that all the moments of the charge distribution function can be expressed through the S matrix only, rather than through the auxiliary matrices defined in Eq. (2).

Before presenting Eq. (5) we shall illustrate its applications with a few simple examples. We start with a single channel case ($n = 1$) [16] with the S matrix depending on time as

$$S(\tau) = e^{i\sigma_3\theta(\tau)/2}\tilde{S}e^{i\sigma_3\theta(\tau)/2}, \tag{11}$$

where $\theta(\tau + \tau_0) - \theta(\tau) = 2\pi$, and $\tilde{S}$ is time independent. This form of the S matrix is equivalent to the phase winding of the reflection amplitudes, $r \to r\exp\{i\theta(\tau)\}$ ($r' \to r'\exp\{-i\theta(\tau)\}$). We note that although such an S matrix could arise from a moving point scatterer it can also arise from a scatterer with a periodic in time Hamiltonian, i.e., formed by two barriers whose heights and positions periodically depend on time. The contribution from the chiral anomaly in Eq. (8) coincides with the number of cycles, $\hat{N} = N$. We concentrate first on a particularly simple time dependence, $\theta(\tau) = 2\pi\tau/\tau_0$. In this case the determinant in Eq. (5) may be easily calculated in the Fourier basis leading to

$$F_N(\lambda) = \prod_k [1 + n_L(\epsilon_{k+N})[1 - n_R(\epsilon_k)](e^{i\lambda} - 1)|t|^2 + [1 - n_L(\epsilon_{k+N})]n_R(\epsilon_k) \times (e^{-i\lambda} - 1)|t|^2]e^{iN\lambda}, \tag{12}$$

where $\epsilon_k = \pi(2k + 1)/(N\tau_0)$ are fermionic frequencies. At small temperature $T \ll 1/\tau_0, V$, this expression simplifies substantially, leading to

$$F_N(\lambda) = (e^{i\lambda}|r|^2 + |t|^2)^N(|r|^2 + e^{-i\lambda}|t|^2)^{N\tau_0 V/(2\pi)}, \tag{13}$$

where V is a voltage applied between the left and right leads, which is chosen to be such that $N\tau_0 V/(2\pi)$ is an integer [9]. In the absence of the external voltage Eqs. (4) and (13) lead to the *binomial* distribution function of the charge transmitted through the adiabatic pump,

$$P_N(Q) = C_N^Q|r|^{2Q}(1 - |r|^2)^{N-Q}, \tag{14}$$

where $C_N^Q \equiv N!/[Q!(N - Q)!]$. Note that only the integer values of the transmitted charge have nonzero probability to be detected. The physical meaning of this expression is that each pumping cycle is associated with an attempt to transfer one electron. The success probability of such an attempt is given by the reflection coefficient $|r|^2$, whereas the probability of failure is $1 - |r|^2$. The above statistics should be compared with the case of the dc voltage applied across the scattering region in the absence of pumping. This case may be obtained from Eq. (13) in the limit $N \to 0$, whereas $N\tau_0 V/(2\pi) \to \tilde{N}$ − integer. We immediately recover the familiar result [9]

$$P_{\tilde{N}}(Q) = C_{\tilde{N}}^Q|t|^{2Q}(1 - |t|^2)^{\tilde{N}-Q}. \tag{15}$$

This distribution is also binomial, however the probability of success is given by the transmission coefficient $|t|^2$. The variances of the transmitted charge for the adiabatic pump and for applied dc voltage coincide and are given by

$$\langle\langle Q^2 \rangle\rangle = |t|^2(1 - |t|^2)N, \tag{16}$$

leading to the maximal noise power in both cases at $|t|^2 = |r|^2 = 1/2$.

The binomial distribution, Eq. (14), was derived above for the simplest time dependence of the form $r(\tau) = r\exp\{2\pi i\tau/\tau_0\}$. As shown in Refs. [9,10] the same result holds for a more general class of "coherent" pumping strategies,

$$e^{i\theta(\tau)} = \frac{e^{2\pi i\tau/\tau_0} - z}{1 - z^* e^{2\pi i\tau/\tau_0}}, \quad (17)$$

where z is a complex number with $|z| < 1$.

Such coherent pumping strategy minimizes the variance of the transmitted charge. Indeed, substituting the S matrix of the form Eq. (11) into Eq. (10) and minimizing $\langle\langle Q^2\rangle\rangle$ with respect to $e^{i\theta(t)}$ [17], one finds that the coherent pumping, Eqs. (11) and (17), leads to the minimal variance given by Eq. (16).

Next we consider a 2×2 scattering matrix with real reflection and transmission amplitudes given by $r = -r' = \cos(2\pi\tau/\tau_0)$ and $t = t' = \sin(2\pi\tau/\tau_0)$, respectively. In this case $U(\tau) = V(\tau) = 1$ leading to $\hat{N} = 0$, and $\tilde{n}$ is the equilibrium distribution function. This is an example of a "pump" which produces only noise and no average current. The noise, however, has nontrivial quantum correlations (see below) and may be studied experimentally. The absence of the average current is useful to increase "noise to current" ratio. One can show that $F_N(\lambda)$ in Eq. (5) is an even function of λ, and is therefore real. As a result one may employ the method of Ref. [11] to compute the determinant of the operator in Eq. (5): one multiplies this operator by its Hermitian conjugate and takes the square root. The resulting operator may be written as $1 + (1 - \tilde{n})\tilde{S}_\lambda^\dagger \tilde{S}_{-\lambda}\tilde{n} + \tilde{n}S_{-\lambda}^\dagger \tilde{S}_\lambda(1 - \tilde{n})$. In the energy representation it has a finite number of the off-diagonal matrix elements and its determinant can be straightforwardly evaluated. This way one obtains

$$F_N(\lambda) = \left[\frac{1 + \cos\lambda}{2}\right]^N. \quad (18)$$

There are three possible values of the transmitted charge in each cycle: $Q = 0$ with the probability $1/2$, and $Q = \pm 1$ with the probability $1/4$ each.

We note that the logarithmic derivative $iV^\dagger(\tau)\partial V(\tau)/\partial\tau$ is analogous to the instantaneous matrix of "voltages" applied to the *incoming* channels. The integral of this quantity can be interpreted as the number of transmission attempts [9]. The different outcomes of such attempts lead to the noise of the pumping current. In general the probability distribution of the transmitted charge is not binomial. If the "voltage" matrix cannot be diagonalized simultaneously with the reflection and transmission matrices the distribution function of the transmitted charge does not factorize into binomial distributions of elementary transmission processes.

In contrast, the matrix $U(\tau)$ corresponds to the *outgoing* channels and enters the final expression (5) only through the chiral anomaly term (8), and therefore contributes to the average current but not to the noise. For example, the pumping cycle of the form $S(\tau) = U(\tau)\tilde{S}$ at zero temperature would produce a noiseless quantized pumping current.

We turn now to the derivation of Eq. (5). To this end we model the leads by a $2n$-component vector of chiral incoming fermions $[\psi_L(x,\tau), \psi_R(x,\tau)]$ and $2n$-component vector of chiral outgoing fermions $[\xi_L(x,\tau), \xi_R(x,\tau)]$. The action for, e.g., the left lead is written as

$$S_L = \int_C d\tau \int_{-\infty}^0 dx\, \overline{\psi}_L(\partial_t + \hat{v}_L\partial_x)\psi_L + \overline{\xi}_L(\partial_t - \hat{v}_L\partial_x)\xi_L, \quad (19)$$

where $\hat{v}_L$ is a diagonal $n \times n$ matrix of the left lead channel velocities. In this expression the time integral runs along the Keldysh contour, C, from $\tau = 0$ to $\tau = N\tau_0$ and then back to $\tau = 0$. The right lead is described by the similar action with the space integral running from $x = 0$ to $x = +\infty$, and the velocity matrix $\hat{v}_R$. Finally the incoming and outgoing channels at $x = 0$ are related by the time-dependent S-matrix operator

$$\xi(0,\tau) = \hat{v}^{1/2}S(\tau)\hat{v}^{-1/2}\psi(0,\tau). \quad (20)$$

The current operator has a form $I = (I_L + I_R)/2$, where

$$I_L(\tau) = [\overline{\psi}_L(0^-)\hat{v}_L\psi_L(0^-) - \overline{\xi}_L(0^-)\hat{v}_L\xi_L(0^-)]. \quad (21)$$

The operator of the charge transmitted in N cycles is given by $Q = \int_0^{N\tau_0} d\tau\, I(\tau)$. Finally, the generating function may be written as

$$F_N(\lambda) = \int D[\psi,\xi]e^{-S_L - S_R + (i/2)\int_C d\tau\,\hat{\lambda}(\tau)I(\tau)}, \quad (22)$$

where $\hat{\lambda}(\tau)$ is equal to λ on the forward and $-\lambda$ on the backward part of the Keldysh contour. The fermion fields in this integral obey the boundary condition, Eq. (20). One has to specify the initial, $\tau = 0$, density matrix, which implicitly defines the Green functions. We fix the occupation numbers in the incoming channels of the left and right leads to be $n_L(\epsilon)$ and $n_R(\epsilon)$ correspondingly, whereas the outgoing channels are supposed to be initially empty in accord with the scattering setup.

The subsequent calculations amount to the evaluation of the Gaussian integral in Eq. (22). To this end we first make the *chiral* gauge transformation of the fermionic fields: $\psi(x,\tau) \to V(\tau)\psi(x,\tau)$ and $\xi(x,\tau) \to U(t)\xi(x,\tau)$. As a result, the boundary condition for the new fermions contains the $\tilde{S}(\tau)$ matrix only and the action acquires an additional time-dependent (matrix) chemical potential term

$$\delta S = \int_C d\tau \int dx\, \overline{\psi}[V^\dagger\partial_\tau V]\psi + \overline{\xi}[U^\dagger\partial_\tau U]\xi. \quad (23)$$

Such a potential term results in the redefinition of the density matrix according to Eq. (7). Importantly, upon the chiral gauge transformation the expression for the current acquires an extra term $I \rightarrow I + 1/(4\pi i) \times \mathrm{Tr}\{\partial U^\dagger(\tau)\sigma_3 U(\tau)/\partial\tau - \partial V^\dagger(\tau)\sigma_3 V(\tau)/\partial\tau\}$ arising from the chiral anomaly [18].

Since the source field, $\hat{\lambda}(\tau)$, is a constant on both branches of the Keldysh contour, one may eliminate the $\int \hat{\lambda} I$ term from the action by the time-independent gauge transformation [9], e.g., $\psi_L \rightarrow e^{i\theta(x-0^-)\lambda/2}\psi_L$ on the forward branch of the contour and $\psi_L \rightarrow e^{-i\theta(x-0^-)\lambda/2}\psi_L$ on the backward branch. Such transformation leads to the change in the phase of the forward scattering amplitude and can be taken into account by a redefinition of the $\tilde{S}$ matrix in the boundary condition, Eq. (20), $\tilde{S} \rightarrow \tilde{S}_{\pm\lambda}$ on the forward (backward) branches, with $\tilde{S}_\lambda$ defined in Eq. (6).

The subsequent steps are straightforward. One integrates out all degrees of freedom except for those which reside directly at the scatterer, $x = 0$. Using the boundary condition, Eq. (20), with $\tilde{S}_{\pm\lambda}$ matrix, one eliminates the incoming degrees of freedom, $\psi(x = 0, \tau)$. The remaining Gaussian integral over the outgoing fermions, $\xi(x = 0, \tau)$, can be straightforwardly evaluated, resulting in the determinant written in Eq. (5). The remaining term, $\exp\{i\hat{N}\lambda\}$, is the contribution from the chiral anomaly, as explained above.

To conclude, we have derived a general expression for the counting statistics of the charge transmitted through a system described by a time-dependent S matrix. The only limitations of our result are the requirements of adiabaticity and the absence of inelastic processes in the scattering region. The absolute minimum of the noise power may be achieved by the coherent pumping strategy, in which case the charge distribution is given by the product of binomial distributions. We point out the major role played by the chiral anomaly contribution to the average transmitted charge.

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