Conductivity of a Clean One-Dimensional Wire

A. Rosch and N. Andrei

Center of Materials Theory, Rutgers University, Piscataway, New Jersey 08854-0849

(Received 22 February 2000)

We study the low-temperature low-frequency conductivity σ of an interacting one-dimensional electron system in the presence of a periodic potential. The conductivity is strongly influenced by conservation laws, which, we argue, need to be violated by at least *two* noncommuting umklapp processes to render σ finite. The resulting dynamics of the slow modes is studied within a memory matrix approach, and we find an exponential increase as the temperature is lowered, $\sigma \sim (\Delta n)^2 e^{T_0/(NT)}$ close to commensurate filling M/N, $\Delta n = n - M/N \ll 1$, and $\sigma \sim e^{(T_0'/T)^{2/3}}$ elsewhere.

PACS numbers: 73.50.Bk, 71.10.Pm, 72.10.Bg

The finite-temperature conductivity of a clean one-dimensional wire [1] is a fundamental and much studied question. Clearly the "bulk" conductivity of a wire in the *absence* of a periodic potential is infinite even at finite temperatures T. In this case the conductance is independent of the length of the wire and is determined by the contacts only. Surprisingly, much less is known about the conductivity in the presence of umklapp scattering induced by a periodic potential. There is not even an agreement whether it is finite or infinite at finite temperatures for generic systems [2–6]. We shall show that the correct answer emerges when all relevant (weakly violated) conservation laws are taken into account. Those conservation laws are exact at the Fermi surface and are violated by umklapp terms away from it. We shall study the associated slow modes by means of a *memory matrix* formalism able to keep track of their dynamics. It will allow us to calculate reliably the low-temperature, low-frequency conductivity.

The topology of the Fermi surface of a 1D metal determines its low-energy excitations. Two well-defined Fermi points exist at momenta $k = \pm k_F$, allowing us to define left and right moving excitations, to be described by $\Psi_{L/R,\sigma=\uparrow\downarrow}$. We shall include in the fields momentum modes extending to the edge of the Brillouin zone, usually omitted in treatments that concentrate on physics very close to the Fermi surface.

The Hamiltonian, including high-energy processes, is

$$H = H_{LL} + H_{\mathrm{irr}} + \sum_{n,m}^{\infty} H^U_{n,m}. \tag{1}$$

H_{LL} is the well-known Luttinger liquid Hamiltonian capturing the low-energy behavior [1],

$$H_{LL} = v_F \int (\Psi^\dagger_{L\sigma} i\partial_x \Psi_{L\sigma} - \Psi^\dagger_{R\sigma} i\partial_x \Psi_{R\sigma}) + g \int \rho(x)^2$$
$$= \frac{1}{2} \int \frac{dx}{2\pi} \sum_{\nu=\sigma,\rho} v_\nu \left(K_\nu (\partial_x \theta_\nu)^2 + \frac{1}{K_\nu} (\partial_x \phi_\nu)^2 \right),$$

v_F is the Fermi velocity, $g > 0$ measures the strength of interactions, and $\rho = \rho_L + \rho_R$ is the sum of the left- and right-moving electron densities. In the second line, we wrote the bosonized [1] version of the Hamiltonian. Here v_σ, v_ρ are the spin and charge velocities, and the interactions determine the Luttinger parameters K_ν with $v_\nu K_\nu = v_F$, $v_\rho/K_\rho = v_F + g/\pi$, $v_\sigma/K_\sigma = v_F - g/\pi$.

The high-energy processes are captured in the subsequent terms which are formally irrelevant at low energies (we consider only systems away from a Mott transition, i.e., away from half filling). Some of them, however, determine the low-frequency behavior of the conductivity at any finite T, since they induce the decay of the conserved modes of H_{LL} (they are "dangerously irrelevant"). We classify these irrelevant terms with the help of two operators which will play the central role in our discussion. The first one is the translation operator P_T of the right- and left-moving fields; the second one, $J_0 = N_R - N_L$, is the difference of the number of right- and left-moving electrons, and is up to v_F, the charge current of H_{LL}:

$$P_T = \sum_\sigma \int dx [\Psi^\dagger_{R\sigma}(-i\partial_x)\Psi_{R\sigma} + \Psi^\dagger_{L\sigma}(-i\partial_x)\Psi_{L\sigma}], \tag{2}$$

$$J_0 = N_R - N_L = \sum_\sigma \int dx (\Psi^\dagger_{R\sigma}\Psi_{R\sigma} - \Psi^\dagger_{L\sigma}\Psi_{L\sigma}). \tag{3}$$

Both P_T and J_0 are conserved by H_{LL}; their importance for transport properties is due to the fact that both stay *approximately* conserved in *any* one-dimensional metal (away from half filling): processes which change J_0 are forbidden *close to the Fermi surface* by momentum conservation. The linear combination $P_0 = P_T + k_F J_0$ can be identified with the total momentum of the full Hamiltonian H and is therefore also approximately conserved.

We proceed to the classification of the formally irrelevant terms in the Hamiltonian. This classification allows us to select all those terms (actually few in number) that determine the current dynamics. H_{irr} includes all terms in $H - H_{LL}$ which commute with both P_T and J_0, such as corrections due to the finite band curvature, due to finite-range interactions and similar terms. We will not need their explicit form.

The umklapp terms $H^U_{n,m}$ $(n, m = 0, 1, \ldots)$ convert n right movers to left movers (and vice versa) picking up

 0031-9007/00/85(5)/1092(4)$15.00

lattice momentum $m2\pi/a = mG$, and do not commute with either P_T or J_0. Leading terms are of the form

$$H^U_{0,m} \approx g^U_{0,m} \int e^{i\Delta k_{0,m}x}(\rho_L + \rho_R)^2 + \text{H.c.}, \quad (4)$$

$$H^U_{1,m} \approx g^U_{1,m} \sum_\sigma \int e^{i\Delta k_{1,m}x}\Psi^\dagger_{R\sigma}\Psi_{L\sigma}\rho_{-\sigma} + \text{H.c.}, \quad (5)$$

$$H^U_{2,m} \approx g^U_{2,m} \int e^{i\Delta k_{2,m}x}\Psi^\dagger_{R\uparrow}\Psi^\dagger_{R\downarrow}\Psi_{L\downarrow}\Psi_{L\uparrow} + \text{H.c.}, \quad (6)$$

with momentum transfer $\Delta k_{n,m} = n2k_F - mG$. A process transferring $n > 1$ electrons with total spin $n_s/2$ pointing in the z direction can be neatly expressed as

$$H^U_{n,m} \approx \frac{g^U_{n,m,n_s}}{(2\pi\alpha)^n} \int e^{i\Delta k_{n,m}x} e^{i\sqrt{2}(n\phi_\rho + n_s\phi_\sigma)} + \text{H.c.}, \quad (7)$$

α being a cutoff, of the order of the lattice spacing. In fermionic variables, the integrand takes the form $\prod_{j=0}^{n/2-1}\Psi^\dagger_{R\uparrow}(x + j\alpha)\Psi^\dagger_{R\downarrow}(x + j\alpha)\Psi_{L\downarrow}(x + j\alpha)\Psi_{L\uparrow}(x + j\alpha)$ (for $n_s = 0$ and even n).

Note, though, that any *single* term H^U_{nm} conserves a linear combination of J_0 and P_T:

$$[H^U_{nm}, \Delta k_{nm}J_0 + 2nP_T] = 0. \quad (8)$$

Indeed, a term of the form (7) would appear in a continuum model *without* umklapp scattering, but with a Fermi momentum $\tilde{k}_F = \Delta k_{nm}/(2n)$. In such a model, $\Delta k_{nm}J_0/(2n) + P_T$ is the total momentum of the system and therefore conserved. The importance of this simple but essential conservation law has, to our knowledge, not been sufficiently realized in previous calculations of the conductivity. Because of this conservation law, a single umklapp term can never induce a finite conductivity. At least two independent umklapp terms are required to lead to a complete decay of the current. Further, two incommensurate umklapp terms suffice to generate the rest.

To calculate the conductivity, it is necessary to keep track of the nearly conserved quantities and their relation to the current. We will develop a description of the slowest variables using the Mori-Zwanzig memory functional [2,7,8]. Approximations within this scheme amount to short-time expansions. In general, the short-time decay of a quantity carries little information on its long-time behavior; this, however, is *not* the case for the slowest variables in the system, where the short-time and hydrodynamic behavior coincide.

To set up the formalism [7], we define a scalar product $(A|B)$ in the space of *operators*,

$$(A(t)|B) \equiv \frac{1}{\beta}\int_0^\beta d\lambda \langle A(t)^\dagger B(i\lambda)\rangle, \quad (9)$$

where we use the usual Heisenberg picture with $A(t) = e^{iHt}Ae^{-iHt}$. We choose a set of "slow" operators $j_1, j_2, \ldots, j_N$ which includes $j_1 = J$, the full current operator. Standard arguments [7] lead to the electric conductivity,

$$\sigma(\omega, T) = \{[\hat{M}(\omega, T) - i\omega]^{-1}\hat{\chi}(T)\}_{11}. \quad (10)$$

Here $\hat{\chi}_{pq} = \beta(j_p|j_q)$ is the matrix of the static $j_p j_q$ susceptibilities (as usually defined), and $\hat{M}$ is the matrix of memory functions given by the projected correlation functions of time derivatives of the slow operators,

$$\hat{M}_{pq}(\omega) = \beta\sum_{r=1}^N \left(\partial_t j_q \,\middle|\, Q\frac{i}{\omega - QLQ}Q \,\middle|\, \partial_t j_r\right)(\hat{\chi}^{-1})_{rp}. \quad (11)$$

The Liouville "super"operator, L, is defined by $LA = [H, A]$, and Q is the projection operator on the space perpendicular to the slowly varying variables j_p:

$$Q = 1 - \sum_{pq}|j_q)\beta(\hat{\chi}^{-1})_{qp}(j_p|. \quad (12)$$

We assumed for simplicity that all j_p have the same signature under time reversal.

The perturbative expansion of the memory matrix $\hat{M}$ is accompanied by factors $1/\omega$ guaranteeing it is always valid at short times. It is also valid for small frequencies provided the slowly evolving degrees of freedom are projected out (by the operator Q). Unlike the conductivity, it is expected to be a smooth function of the coupling constants which can be perturbatively evaluated.

We first consider a situation where some linear combinations of the j_p are conserved by H, in which case an infinite conductivity is expected. We introduce $\mathcal{P}_c$, the projection operator on the space of conserved currents, and carry out the required matrix inversion to find

$$\sigma(\omega \to 0, T > 0) = \sigma_{\text{reg}}(\omega, T) + i\frac{(\hat{\chi}\hat{\chi}_c^{-1}\hat{\chi})_{11}}{\omega + i0}, \quad (13)$$

where $\hat{\chi}_c^{-1} = \mathcal{P}_c(\mathcal{P}_c\hat{\chi}\mathcal{P}_c)^{-1}\mathcal{P}_c$. Within any simple (short-time) approximation, $\sigma_{\text{reg}}(\omega, T)$, as defined above, is regular (this approximation fails, e.g., if some conserved current $\tilde{j}$ is not included in $j_1, \ldots, j_N$). Hence, the Drude weight $D(T)$ is finite at finite temperatures, $\text{Re}\sigma(\omega \to 0) = 2\pi D(T)\delta(\omega) = \pi(\hat{\chi}\hat{\chi}_c^{-1}\hat{\chi})_{11}\delta(\omega)$. It is determined by the "overlap" of the physical current operator J with the conserved quantities χ_{1s}, s labeling the conserved currents. Remarkably, our perturbative approximation is in accord with an exact inequality [5] for the Drude weight, $D(T) \geq \frac{1}{2}(\hat{\chi}\hat{\chi}_c^{-1}\hat{\chi})_{11}$. Note that $\hat{\chi}$ can be calculated to an arbitrary degree of precision around a Luttinger liquid and that the lower bound can be improved by including more conserved quantities [5].

Now consider the more realistic situation where the previously conserved currents decay slowly (via umklapp processes), in which case a finite conductivity is expected. We restrict ourselves to the two-dimensional space spanned by $v_F J_0$ and P_T, which we argue have the longest decay rate and dominate the transport. Here we approximate $J \approx v_F J_0$ to keep the presentation simple. This affects

only the high-frequency behavior of the conductivity [3]. There is a large number of other nearly conserved quantities. For example, $H_{LL} + H_{21}^U$, the relevant low-energy model close to half filling, is integrable and therefore is characterized by an *infinite* number of conservation laws. We can, however, neglect them at low T if our initial model is not integrable, expecting that practically all conservation laws are destroyed by (formally irrelevant) terms *close to the Fermi surface* leading to decay rates proportional to some power of T. This is to be compared to J_0 and P_T which commute with *all* scattering processes at the Fermi surface, leading to exponentially large lifetimes.

We now proceed to calculate the memory matrix. To leading order in the perturbations, we can replace L in (11) by $L_{LL} = [H_{LL}, .]$ [9], since $\partial_t v_F J_0$ and $\partial_t P_T$ are already linear in $g_{n,m}^U$. As $L_{LL} P_T = L_{LL} J_0 = 0$, there is no contribution from the projection operator Q. The memory matrix takes the form

$$\hat{M} \approx \sum_{nm} M_{nm}(\omega, T) \begin{pmatrix} v_F^2 (2n)^2 & -2n v_F \Delta k_{nm} \\ -2n v_F \Delta k_{nm} & (\Delta k_{nm})^2 \end{pmatrix} \hat{\chi}^{-1},$$

where

$$\hat{\chi} \approx \begin{pmatrix} 2v_F/\pi & 0 \\ 0 & \frac{\pi T^2}{3}\left(\frac{1}{v_\rho^3} + \frac{1}{v_\sigma^3}\right) \end{pmatrix}, \tag{14}$$

$$M_{nm} \equiv (g_{nm}^U)^2 M_n(\Delta k_{n,m}, \omega) \equiv \frac{\langle F; F\rangle_\omega^0 - \langle F; F\rangle_{\omega=0}^0}{i\omega}.$$

Here, $F = [J_0, H_{nm}^U]/(2n)$ (for simplicity we drop the indices n, m on F), and $\langle F; F\rangle_\omega^0$ is the retarded correlation function of F calculated with respect to H_{LL}.

The memory function M_2 of the $4k_F - Q$ process H_{21}^U was calculated by Giamarchi [2] (not considering the matrix structure of $\hat{M}$ required by the conservation laws). Higher umklapps are considered in [3]. For $n_s = 0$ and even n the memory function due to the term (7) can be analytically calculated:

$$\begin{aligned} M_n(\Delta k, \omega) = {} & \frac{2\sin 2\pi K_\rho^n}{\pi^4 \alpha^{2n-2} v_\rho} \left[\frac{2\pi\alpha T}{v_\rho}\right]^{4K_\rho^n - 2} \frac{1}{i\omega} \\ & \times [B(K_\rho^n - iS_+, 1 - 2K_\rho^n) B(K_\rho^n - iS_+, 1 - 2K_\rho^n) - B(K_\rho^n - iS_+^0, 1 - 2K_\rho^n) B(K_\rho^n - iS_+^0, 1 - 2K_\rho^n)] \\ \approx {} & \frac{\alpha^{2-2n}}{\pi^2 \Gamma^2(2K_\rho^n) v_\rho T} \left(\frac{\alpha \Delta k}{2}\right)^{4K_\rho^n - 2} e^{-v_\rho \Delta k/(2T)}, \end{aligned}$$

where $K_\rho^n = (n/2)^2 K_\rho$, $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$, and $S_\pm = (\omega \pm v_\rho \Delta k)/(4\pi T)$, $S_\pm^0 = S_\pm(\omega = 0)$. The last line is valid for $\omega = 0$ and $T \ll v_\rho \Delta k$.

The origin of the exponential factor is as follows: processes involving momentum transfer Δk are associated with initial and final states of energies $v|\Delta k|/2$, which are exponentially suppressed. If only charge degrees of freedom are involved, $v = v_\rho$, otherwise $v = \min(v_\sigma, v_\rho) = v_\sigma$ [9]. For $T \ll v_\sigma \Delta k_{nm}$, $n_s > 0$ and $\omega = 0$, we have

$$M_n(\Delta k) \sim \frac{(\alpha T/v_\rho)^{n^2 K_\rho - 1} (\alpha \Delta k)^{n_s^2 K_\sigma - 2}}{\Gamma^2(n_s^2 K_\sigma/2) v_\sigma^2 \alpha^{2n-3}} e^{-v_\sigma \Delta k/(2T)}, \tag{15}$$

while, for $T \gg v_\rho \Delta k_{nm}$, $M_n \sim T^{n^2 K_\rho + n_s^2 K_\sigma - 3}$.

Using the above expressions with only one umklapp term leads to a finite Drude weight [cf. Eq. (13)],

$$D(T) \approx \frac{v_\rho K_\rho}{\pi} \frac{1}{1 + T^2 \frac{2\pi^2 n^2 K_\rho}{3(v_\rho \Delta k_{nm})^2}\left(1 + \frac{v_\rho^3}{v_\sigma^3}\right)}, \tag{16}$$

in accord with the observation that one process H_{nm}^U is not sufficient to degrade the current.

Only in the presence of a second incommensurate process $H_{n'm'}^U$ is the dc conductivity finite:

$$\sigma(T) = \frac{(\Delta k_{nm})^2/M_{n'm'} + (\Delta k_{n'm'})^2/M_{nm}}{\pi^2 (n \Delta k_{n'm'} - n' \Delta k_{nm})^2}. \tag{17}$$

Note that the *slowest* process determines the low-T conductivity. The frequency and temperature dependence of the conductivity in the case of two competing umklapp terms is shown in Fig. 1.

The commensurate situation $\Delta k_{nm} = 0$ requires extra considerations. Whether the dominant scattering process H_{nm} will completely relax the current J depends according to (13) on the overlap χ_{JP_T} ($[P_T, H_{nm}] = 0$). Using the continuity equation for the charge, χ_{JP_T} can be related to

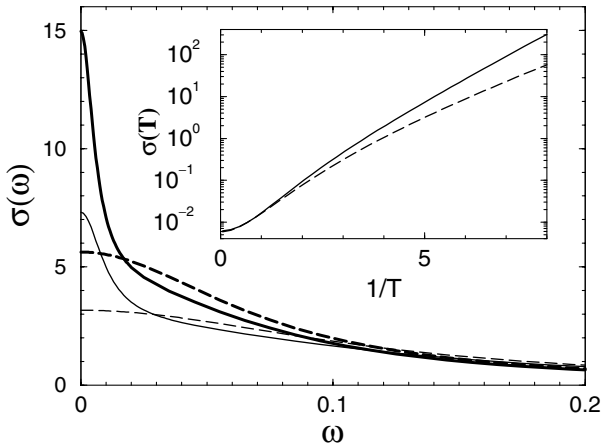

FIG. 1. The low-frequency behavior of $\sigma(\omega)$ in the presence of two umklapp terms for two different T. The dashed lines are the result one obtains in conventional perturbation theory neglecting [2] the matrix structure of $\hat{M}$ and the related conservation laws. ($g_{20} = g_{21} = 1$, $K_\rho = 0.7$, $K_\sigma = 1.3$, $\Delta k_{21} = -1.5\Delta k_{20}$, thick lines $T = 0.2$, thin lines $T = 0.18$, ω and T measured in units of $v_\rho \Delta k_{20}$.) Note that two time scales appear, each describing the scale on which the associated conservation law is violated. The inset displays the T dependence of $\sigma(\omega = 0)$.

the deviation $\Delta\rho = 2\Delta n/a$ of the electron density from commensurate filling with the remarkable identity $\chi_{JP_T} = 2\Delta n/a + O(e^{-\beta\epsilon_F})$. In a 3D lattice of 1D wires, Δn is fixed by charge neutrality and is T independent; in a single wire with contacts, Δn varies at low T with $\Delta n(T) \sim T^2/(mv^3)$, where the mass m is a measure of the breaking of particle-hole symmetry, e.g., due to a band curvature $k^2/2m$. In this case, it is important to replace $\Delta k_{nm} = 0$ in Eqs. (16) or (17) by $G\Delta n(T)$.

Which of the various scattering processes will eventually dominate at lowest T? At intermediate temperatures, certainly low-order (small n) scattering events win, being less suppressed by Pauli blocking. At lower temperature, the exponential factors in (15) prevail and the processes with the smallest Δk_{nm} are favored. We first analyze the situation close to a commensurate point $k_F \approx GM_0/(2N_0)$. The *two* dominant processes are $H^U_{N_0M_0}$ with $\Delta k_{N_0M_0} \approx 0$ and $H^U_{N_1M_1}$ with $\Delta k_{N_1M_1} = \pm G/N_0$ (or $N_1M_0 = \pm 1 \bmod N_0$). The integer N_1 of order N_0, $N_1 = \gamma_1 N_0$, depends strongly on the precise values of N_0 and M_0. We thus find that the dc conductivity at low T is *largest* close to commensurate points with,

$$\sigma\big(k_F \approx GM_0/(2N_0)\big) \sim [\Delta n(T)]^2 \exp[\beta v G/(2N_0)], \tag{18}$$

but $\sigma \sim T^{-N_0^2K_\rho - (N_0 \bmod 2)^2 K_\sigma + 3}$ if the density is exactly commensurate with $|\Delta n(T)| < e^{-\beta G v/(4N_0)}$.

To estimate the conductivity at a typical "incommensurate" point or at commensurate points at temperatures not too low, we have to balance algebraic and exponential suppression in (15) by minimizing $-\beta v G/(2N) + (\gamma_1 N)^2 K \log[T]$ in a saddle-point approximation to the sum over all umklapp processes in $\hat{M}$. Up to logarithmic corrections, we obtain $N^3_{\max} \sim \beta v G/(\gamma_1)^2$ and therefore, for a "typical" incommensurate filling,

$$\sigma_{\text{typical}} \sim \exp[c(\beta v G)^{2/3}], \tag{19}$$

where c is a number depending logarithmically on T. At present we cannot rule out that various logarithmic corrections sum up to modify the power law in the exponent. We argue, however, that, due to the exponential increase (18) of σ at commensurate fillings with exponents proportional to $1/N_0$, the conductivity at small T at any incommensurate point is smaller than any exponential (but is larger than any power since any single process is exponentially suppressed). In Fig. 2 we show schematically the conductivity as a function of filling becoming more and more "fractal-like " for lower T.

Can the effects we predict be seen experimentally? The complicated structures as a function of filling shown in Fig. 2 are not observable in practice as they occur only at exponentially large conductivities. The T dependence of the conductivity at intermediate temperatures, however, should be accessible, e.g., by comparing the conductivities of clean wires of different length. Perhaps more importantly, it is straightforward to apply our method to a large number of other relevant situation, e.g., close to a Mott transition or in the presence of 3D phonons, as we will discuss in a forthcoming paper.

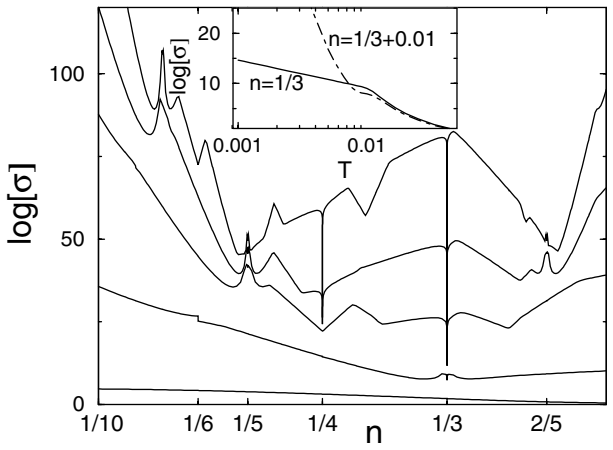

FIG. 2. Schematic plot of $\log[\sigma]$ as a function of the filling $n = 2k_F/G$ for various temperatures ($vG\beta = 30, 100, 300, 500, 800$) based on an asymptotic [Eq.(15)] evaluation of (10). Near commensurate fillings n_c, the conductivity is strongly enhanced at low temperatures but drops at $n = n_c$. The inset displays the T dependence of σ for $n = 1/3$ and a filling very close to $1/3$ (dashed line).

We thank R. Chitra, A. J. Millis, E. Orignac, A. E. Ruckenstein, S. Sachdev, and P. Wölfle for helpful discussions. Part of this work was supported by the A. v. Humboldt Foundation and NSF Grant No. DMR9632294 (A. R.).

[1] J. Sólyom, Adv. Phys. **28**, 209 (1979); V. J. Emery in *Highly Conducting One-Dimensional Solids,* edited by J. Devreese *et al.* (Plenum, New York, 1979), p. 247.
[2] T. Giamarchi, Phys. Rev. B **44**, 2905 (1991).
[3] T. Giamarchi and A. J. Millis, Phys. Rev. B **46**, 9325 (1992).
[4] S. Fujimoto and N. Kawakami, J. Phys. A **31**, 465 (1998); X. Zotos, Phys. Rev. Lett. **82**, 1764 (1999); H. Castella, X. Zotos, and P. Prelovšek, Phys. Rev. Lett. **74**, 972 (1995).
[5] X. Zotos, F. Naef, and P. Prelovšek, Phys. Rev. B **55**, 11 029 (1997).
[6] S. Kirchner *et al.,* Phys. Rev. B **59**, 1825 (1999); S. Sachdev and K. Damle, Phys. Rev. Lett. **78**, 943 (1997); V. V. Ponomarenko and N. Nagaosa, Phys. Rev. Lett. **79**, 1714 (1997); A. A. Odintsov, Y. Tokura, and S. Tarucha, Phys. Rev. B **56**, 12729 (1997); M. Mori, M. Ogata, and H. Fukuyama, J. Phys. Soc. Jpn. **66**, 3363 (1997); K. Le Hur, Phys. Rev. B (to be published).
[7] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, Massachusetts, 1975).
[8] W. Götze and P. Wölfle, Phys. Rev. B **6**, 1226 (1972).
[9] For large Δk_{nm}, corrections from H_{irr} like band-curvature terms are important, we neglect them here. Their inclusion leads again to exponential suppression in Eq. (15) with modified numerical factors.