Order Out of Disorder in a Gas of Elastic Quantum Strings in 2 + 1 Dimensions

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A limiting case of a dynamical stripe state which is of potential significance to cuprate superconductors is considered: a gas of elastic quantum strings in 2 + 1 dimensions, interacting merely via a hard-core condition. It is demonstrated that this gas always solidifies, by a mechanism which is the quantum analog of the entropic interactions known from soft condensed matter physics.

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The analysis of systems of quantum particles has been traditionally the focus point of quantum many body theory. On the other hand, much less is known about systems composed of extended objects. Here I will analyze one of the simplest examples of such a system: a gas of quantum strings with finite line tension, embedded in (2 + 1)-dimensional space-time. A motivation to study this problem is found in the context of the cuprate stripes [1]. It is popular to view these stripes as preformed linelike textures which can either order in a regular pattern or stay in a disordered state due to strong quantum fluctuations. The question arises whether it is possible to quantum melt a system of completely intact, infinitely long stripes. Even in this limit the stripes themselves can still execute quantum meandering motions and a consensus has been growing that a single stripe is similar to a quantum string with finite line tension [2–6]. I define the ideal string gas as the low density limit where the width of the strings can be neglected, while the strings interact only via the requirement that they cannot intersect [7]. This is obviously the limit where quantum kinetic energy is most important. I will show that in 2 + 1 dimensions even in this limit this string system turns into a solid at zero temperature. This solidification is driven by the quantum-mechanical analog of the entropic interactions known from statistical mechanics. In a system with steric interactions between its constituents, entropy is paid at collisions in the classical system and kinetic energy in the quantum system. This causes an effective repulsion and these "quantum entropic" interactions dominate to such an extent in the string gas that they cause it to solidify always.

In the path-integral representation, a quantum-mechanical problem of interacting particles becomes equivalent to a statistical physics problem of interacting elastic lines ("world lines"). Likewise, the quantum string gas becomes equivalent to the statistical physics problem of a stack of elastic membranes ("world sheets") which do not interact except for the requirement that the membranes do not intersect. A seminal contribution in the study of entropic interactions in classical systems composed of extended entities is the analysis by Helfrich [8] of a system of extrinsic curvature membranes in 3D, interacting only via an excluded volume constraint. I will illustrate this method in the quantum context by analyzing the hard-core Bose gas in (1 + 1)D, which is closely related to Helfrich's extrinsic curvature membranes in 3D. The string gas will turn out to be a straightforward, but nontrivial, extension of the Bose gas: different from the latter, the quantum entropic interactions of the string gas are driven by long wavelength fluctuations.

To acquire some insights in the Helfrich method in the context of quantum mechanics, consider the familiar problem of hard-core (but otherwise noninteracting) bosons in (1 + 1)D. This is solved by mapping onto a noninteracting spinless-fermion gas. Although mathematically trivial, this problem does exhibit the conceptual ambiguity associated with Luttinger liquids [9]. On the one hand, it is clearly a gas of particles characterized by a kinetic scale E_F, while at the same time the long-wavelength density-density correlator exhibits the algebraic decay characteristic for a harmonic crystal in (1 + 1)D: $\langle n(x)n(0)\rangle \sim \cos(2k_F x)/x^2$. The concept of entropic interaction offers a simple explanation.

The hard-core Bose gas at zero temperature corresponds with the statistical physics problem of a gas of nonintersecting elastic lines embedded in 2D space-time [10], which are directed along the time direction. The space-like displacement of the ith world line is parametrized in terms of a field $\phi_i(\tau)$ (τ is imaginary time) and the partition function is

$$Z = \Pi_{i=1}^{N} \Pi_\tau \int d\phi_i(\tau) e^{-(S/\hbar)},$$
$$S = \int d\tau \sum_i \frac{M}{2} (\partial_\tau \phi_i)^2, \tag{1}$$

supplemented by the avoidance condition,

$$\phi_1 < \phi_2 < \cdots < \phi_N . \tag{2}$$

The hard-core condition Eq. (2) renders this to be a highly nontrivial problem. Helfrich considered the related classical problem of a stack of linearized and directed extrinsic curvature membranes embedded in 3D space. Although this is a higher dimensional problem, the action depends on double derivatives instead of the single derivatives in Eq. (1), $(\partial_\mu \phi)^2 \to (\partial_\mu^2 \phi)^2$, and it follows from power-counting that this problem is equivalent to the hard-core Bose gas in the present context. In order to determine the "entropic" elastic modulus at long wavelength Helfrich introduced the following construction. Assume that the long-wavelength modulus B_0 is finite. For the Bose gas this implies that the long-wavelength action is that of a (1 + 1)D harmonic solid:

$$S_{\rm eff} = \frac{1}{2} \int d\tau \int dx [\rho(\partial_\tau \psi)^2 + B_0 (\partial_x \psi)^2], \qquad (3)$$

where $\psi(x, \tau)$ is a coarse grained long-wavelength displacement field, $\rho = M/d$ is the mass density, and d is the average inter-world-line distance ($n = 1/d$ is the density). Obviously, for finite B_0 fluctuations are suppressed relative to the case that B_0 vanishes and this cost of kinetic energy in the quantum problem (entropy in the classical problem) raises the free energy. Define this "free-energy of membrane joining" as

$$\Delta F(B_0) = F(B_0) - F(B_0 = 0). \qquad (4)$$

At the same time, by general principle it has to be that the "true" long-wavelength modulus B in the x direction should satisfy (V is the volume)

$$B = d^2 \frac{\partial^2 [\Delta F(B_0)/V]}{\partial d^2}. \qquad (5)$$

In case of the steric interactions, the only source of long-wavelength rigidity in the space direction is the fluctuation contribution to ΔF. This means that $B_0 = B$ and B can be self-consistently determined from the differential equation [Eq. (5)]. In fact, the only ambiguity in this procedure is the choice for the short distance cutoff for the fluctuations in the x direction, which is expected to be proportional to the distance between the world lines, $x_{\rm min} = \eta d$. The shortcoming of the method is that mode couplings are completely neglected and this is not quite correct since the outcomes do depend crucially on short-wavelength fluctuations. However, it appears [11] that these effects can be absorbed in the nonuniversal "fudge factor" η, giving rise to changes in numerical prefactors without affecting the dependence of B on the dimensionful quantities in the problem.

The free energy difference for the Bose gas [Eq. (4)] is easily computed from the Gaussian action [Eq. (3)] and expanding up to leading order in $\lambda = (\sqrt{B}\tau_0)/(\sqrt{\rho}d)$ (τ_0 is the cutoff time), becoming small in the low density limit,

$$\frac{\Delta F}{V} = \frac{\pi \hbar}{4\eta^2} \sqrt{\frac{B}{M}} \frac{1}{d^{3/2}} + O(\lambda^2). \qquad (6)$$

Inserting Eq. (6) on the right-hand side of the self-consistency equation [Eq. (5)] and solving the differential equation up to leading order in λ yields

$$B = \frac{9\pi^2}{\eta^4} \frac{\hbar^2}{Md^3}. \qquad (7)$$

It is easily checked that this corresponds with the elasticity modulus appearing in the bosonized action of the hard-core boson problem, taking $\eta = \sqrt{6}$. Hence, the spacelike rigidity of the Bose gas at long wavelength can be understood as a consequence of entropic interactions living in Euclidean space-time.

Let us now turn to the string-gas problem. In fact, the string gas in (2 + 1)D is related to the hard-core Bose gas in (1 + 1)D: the latter can be viewed as the *compactified* version of the former. Imagine that the hard-core Bose gas lives actually in (2 + 1)D where the additional dimension y is rolled up to a cylinder with a compactification radius R_y of order of the lattice constant a, while the bosons are spread out in elastic strings wrapped around the y axis. Let R_y go to infinite. This has the effect that the embedding space becomes (2 + 1)-dimensional, while the boson world lines spread out in string world sheets. This "directed string-gas" is not yet the one of interest, since the world sheets are not directed only along the imaginary time directions (as required by quantum mechanics) but also in the x-y plane (Fig. 1a). The difficulty is that in the string gas dislocations can occur (Fig. 1b), and if these proliferate they will destroy the generic long range order of the directed string gas. However, two objections can be raised against a dislocation mediated quantum melting. The first objection involves a further specification: already a single string tends to acquire spontaneously a direction, if it is regularized on a *lattice* (like the stripes). As pointed out by Eskes *et al.* [2], the reason is that "overhangs" such as in Fig. 1b are events where transversal fluctuations are surpressed, relative to those around directed configurations. The second argument is more general. It is a classic result [7,12] that at any finite temperature dislocations proliferate in the string gas. However, in the presence of a finite range interaction of any strength the Kosterlitz-Thouless transition will occur at a finite temperature. Hence, by letting this interaction to become arbitrarily weak, a $T = 0$ transition can be always circumvented.

When dislocations can be excluded the directed string gas remains and this is just the decompactified Bose gas.

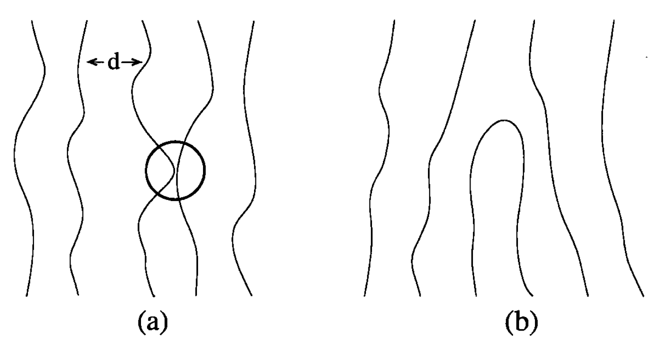

FIG. 1. A typical spacelike configuration in the directed string gas (a), including a collision of the type driving the quantum-entropic interactions. In the string gas dislocations (b) do not proliferate, and it is therefore equivalent to the directed gas.

In Euclidean space-time it corresponds with a sequentially ordered stack of elastic membranes. Orienting the world sheets in the y, τ planes, the action becomes in terms of the displacement fields $\phi_i(y, \tau)$ describing the motion of the strings in the x direction:

$$Z = \Pi_{i=1}^{N} \Pi_{y,\tau} \int d\phi_i(y, \tau) e^{-(\tilde{S}/\hbar)},$$
$$S = \int d\tau\, dy \sum_i \left[\frac{\rho_c}{2} (\partial_\tau \phi_i)^2 + \frac{\Sigma_c}{2} (\partial_y \phi_i)^2 \right], \tag{8}$$

again supplemented by the avoidance condition [Eq. (2)]. In Eq. (8), ρ_c is the mass density and Σ_c is the string tension, such that $c = \sqrt{\Sigma_c/\rho_c}$ is the velocity. In the remainder I choose a lattice regularization with lattice constant a, such that the average string-string distance is $d = a/n$ (n is the density), and UV momenta and frequency cutoffs on a single string $q_0 = \pi/a$ and $\omega_0 = cq_0$, respectively.

Turning to the Helfrich method, the effective long-wavelength action is written as

$$S_{\text{eff}} = \frac{1}{2} \int d\tau\, dx\, dy [\rho(\partial_\tau \psi)^2 + B(\partial_x \psi)^2 + \Sigma(\partial_y \psi)^2], \tag{9}$$

with $\psi(x, y, \tau)$ as the coarse grained long-wavelength displacement field, while $\Sigma = \Sigma_c/d$, $\rho = \rho_c/d$, and B has to be determined. From the action Eq. (9) it follows that the free energy difference Eq. (4) is

$$\frac{\Delta F}{V} = -\frac{\hbar c}{8\pi^2} \int_0^{q_0} dq^2 \int_0^{\pi/(\eta d)} dq_x \ln\left[\frac{\Sigma q^2}{\Sigma q^2 + B q_x^2} \right]. \tag{10}$$

This integral is easily solved analytically and expanding in the small parameter $\lambda = (\sqrt{B}a)/(\sqrt{\Sigma}\eta d)$,

$$\frac{\Delta F}{V} = \frac{\pi \hbar c}{24 \eta^3 \Sigma_c} \left(\frac{B}{d^2} \right) \left(\frac{5}{3} + \ln\left[\frac{\eta^2 \Sigma_c}{a^2} \frac{d}{B} \right] \right) + O(\lambda^4). \tag{11}$$

The free-energy difference is proportional to B/d^2 except for a logarithmic "correction" $\sim (B/d^2)\ln(d/B)$. Since B tends to zero in the low density limit, it is actually this logarithmic correction which determines the low density asymptote of the differential equation which is obtained after substition of Eq. (11) in the self-consistency condition Eq. (5). The physical meaning of this logarithm will be discussed later.

The differential equation determining the fluctuation induced modulus B becomes

$$f(d) = -C_0 \frac{\partial^2}{\partial d^2} \left[f(d) \ln[C_1 d f(d)] + \frac{5}{3} \right], \tag{12}$$

where $f(d) = B/d^2$ and $C_0 = (\pi \hbar c)/(24 \eta^3 \Sigma_c), C_1 = a^2/(\eta^2 \Sigma_c)$. Equation (12) can be simplified using the ansatz $f(d) = \exp[-\Phi(d)]$. It is easy to see that for large d the second derivative terms $\sim \partial_d^2 \Phi$ can be neglected relatively to the first derivative terms ("quasiclassical approximation"). Neglecting the other terms which do not contribute in the low density asymptote [including the one derived from the "5/3" term in Eq. (12)] Φ obeys asymptotically the simple differential equation,

$$(\Phi - 2)\left(\frac{\partial \Phi}{\partial d} \right)^2 = \frac{1}{C_0}, \tag{13}$$

and it follows that $\Phi(d) \sim d^{2/3}$. The full expression for the induced modulus is up to leading order in the density,

$$B = A d^2 e^{-\eta(54/\pi)^{1/3} 1/\mu^{1/3}}, \tag{14}$$

where A is an integration constant and μ is the "coupling constant" for the string gas,

$$\mu = \frac{\hbar}{\rho c d^2}. \tag{15}$$

Equations (14) and (15) represent my central result.

What is the significance of this result? Most importantly, it demonstrates that in parallel with the hard-core Bose gas (and Helfrich's membranes), the string gas is characterized by a fluctuation induced elastic modulus at long wavelength which will be small but finite even at low density. This modulus B appears in the action Eq. (9) which describes an elastic manifold covering $(2 + 1)$D space-time. Equation (14) describes the counterintuitive fact that, upon increasing the kinetic energy of a single string, the rigidity of this medium is actually increasing. The parameter μ is the dimensionless quantity measuring the importance of quantum fluctuations [13]. In order to prohibit diverging fluctuations on the lattice scale, μ should be less than one, while the classical limit is approached when $\mu \to 0$. According to Eq. (14), B depends on μ in a stretched exponential form, such that B increases when μ is increasing. Since quantum dislocation melting is prohibited, the string gas is always a solid, and this solid becomes more rigid when the microscopic quantum fluctuations become more important. This might appear as less surprising when the (directed) string gas is viewed as a decompactified Bose gas. On the one hand, the larger internal dimensionality of the world sheets as compared to the world lines weakens the "quantum-entropic" interactions, but the enlarged overall dimensionality causes the algebraic long range order of the $(1 + 1)$D Bose gas to become the true long range order of the $(2 + 1)$D string gas.

The mechanism behind the quantum entropic interaction is actually different from the one in the Bose gas. In the Bose gas it builds up at short wavelengths, while in the string gas it is driven by the long-wavelength fluctuations living on the strings. An alternative, more intuitive, understanding is available for the Bose-gas result, Eq. (7). This is based on the simple notion that every time membranes/world-lines collide an amount of entropy $\sim k_B$ is paid because the membranes cannot intersect [8,10,14]. Hence, these collisions raise the free energy of the system and this characteristic free energy cost $\Delta F_{\rm coll} \sim k_b T n_{\rm coll}$. The density of collisions $n_{\rm coll}$ is easily calculated: for the world lines, the mean-square transversal displacement as a function of (timelike) arc-length increases like $\langle(\phi(\tau) - \phi(0))^2\rangle = (\hbar/M)\tau$. The characteristic time τ_c it takes for one collision to occur is obtained by imposing that this quantity becomes of order d^2 and a characteristic collision energy scale is obtained $E_F \sim \hbar/\tau_c \sim (\hbar^2/M)n^2$. E_F is of course the Fermi energy: it is the scale separating a regime where world lines are effectively isolated ($E > E_F$, free particles) from one dominated by the collisions ($E < E_F$, Luttinger liquid). The induced modulus follows from naive coarse graining: E_F is the characteristic energy associated with density change, while d is the characteristic length. Therefore, $B \sim E_F/d$, reproducing the result Eq. (7), within a prefactor of order unity.

For the string gas this procedure yields a simple exponential instead of the stretched exponential [Eq. (14)]. The mean-square transversal displacement now depends logarithmically on the world sheet area A: $\langle[\Delta\phi(A)]^2\rangle = \hbar/(\rho c)\ln(A)$. Demanding this to be equal to d^2, the degeneracy scale follows immediately. The characteristic world sheet area A_c for which on average one collision occurs is given by $\hbar/(\rho c)\ln(A_c) \simeq d^2$, where $A_c = c^2\tau_c^2/a^2$ in terms of the collision time τ_c. It follows that $\tau_c \simeq (a/c)e^{1/2\mu}$ and the "Fermi energy" of the string gas is of order $E_F^{\rm str} = \hbar/\tau_c \simeq (\hbar c/a)\exp(-1/2\mu)$ and thereby $B \sim \exp(-1/\mu)$. In fact, the same μ dependence is obtained from Helfrich's method if the logarithm in Eq. (11) is neglected. Hence, this collision picture misses entirely the origin of the quantum-entropic repulsions in the string gas. The reason becomes clear by inspecting the origin of the logarithmic term in the integrations leading to Eq. (11). Cutting off the smallest allowed momenta in the x, τ directions by $q_{\rm min}$ one finds that $\ln[(\eta^2\Sigma_c d)/(a^2 B)] \rightarrow -\ln[(a^2 B)/(\eta^2\Sigma_c d) + a^2 q_{\rm min}^2]$, and this is unimportant for any finite $q_{\rm min}$ in the low density limit. Therefore, the logarithm and thereby the induced modulus are driven by the long-wavelength fluctuations on the strings, and these are not considered in the collision point picture.

In summary, I have analyzed the fluctuation induced interactions in the "ideal" gas of elastic quantum strings in $(2 + 1)$D. A novelty is that in this system the induced elasticity is due to long-wavelength fluctuations, qualitatively different from the short distance physics of the Bose gas. It remains to be seen if these interactions are of relevance in real physical systems. On the one hand, these are rather weak and easily overwhelmed by direct interactions [15]. However, direct string-string interactions which decay exponentially are generic, and in this case the induced interactions can dominate at sufficiently low density because of their stretched exponential dependence on density: in principle the induced interactions can be physical observables. The immediate relevance of my findings lies elsewhere. The strings considered here are idealizations of the stripes but these idealizations are nevertheless close to a popular way of viewing these matters. I have demonstrated that in the absence of zero temperature stripe long range order [16] it has to be that these ideal stripes are broken up in one or the other way.

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