Holtsmark Distributions in Point-Vortex Systems

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The statistics of uncorrelated point vortices in a plane is studied analytically and numerically. Theoretical distributions are obtained with the general method developed by Holtsmark [Ann. Phys. **58**, 577 (1919)] and Chandrasekhar [Rev. Mod. Phys. **15**, 1 (1943)]. They are found to agree with the results of numerical tests. Randomly placed Euler vortices have nearly Gaussian velocity distributions and Lorentzian distributions of the velocity difference. Statistics of other types of point vortices is essentially non-Gaussian.

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A number of fluid models in hydrodynamics and in plasma physics admit singular solutions in the form of two-dimensional, propagating point vortices. To a first approximation they model turbulent vortex patches on the scale of the correlation length. This suggests that the investigation of point-vortex systems could contribute to the understanding of turbulence.

A large part of the work on the statistics of point-vortex systems is based on a seminal paper by Onsager [1] and deals mainly with thermal mean-field equilibrium states. It investigates either the statistics of the coordinates of the vortices or that of the Fourier coefficients of the vorticity field [2–4]. A review of these researches is given in [5]. The thermodynamic theory of point-vortex systems is quite complex and still controversial. By introducing thermodynamic functions (structure function, entropy, Helmholtz free energy etc.), one needs to prove the existence of an extensive thermodynamic limit. Such a proof is a highly nontrivial matter [6,7]. On the other hand, even in the absence of a thermodynamic limit a statistical approach to point-vortex systems is still justified since rigorous limiting probability densities do exist. This point of view has been advocated by Cambell and O'Neil [6] and Kiessling [8], who state that, although formally defined thermodynamic integrals may not be (uniformly) convergent, the statistical mechanics of a finite N-particle system makes perfect sense.

In this paper, we treat a 2D vortical flow as a gas (ergodic ensemble) of point vortices whose coordinates are uncorrelated. Following Refs. [6,8], the properties of this gas are analyzed from first principles, without introducing *ab initio* thermodynamic functions. We investigate the distributions of velocity and velocity difference fields. These fields are of fundamental interest. According to experiments the velocity distribution in turbulence typically has a nearly Gaussian shape, while the velocity difference distribution is nearly Lorentzian (Cauchy) [9]. Explanation of these phenomenona is still controversial.

In point-vortex models, the distribution of vorticity is treated as a collection of point vortices with strengths κ_j. The corresponding flow in the (x, y) plane is

$$\mathbf{v}(\mathbf{r}) = \sum_j \kappa_j \mathbf{e}_z \times \nabla G(\mathbf{r} - \mathbf{r}_j), \tag{1}$$

which means that the velocity $\mathbf{v}_i$ of the ith vortex at $\mathbf{r} = \mathbf{r}_i$ is induced by all other vortices. In Eq. (1), $\mathbf{e}_z$ is the unit vector in the direction perpendicular to the plane, and G is the Green's function of the appropriate differential operator that relates the (generalized) vorticity to the streaming potential. The above equation represents, in fact, a large number of different physical models. The most well known of these models is 2D hydrodynamics (Euler) [10], where the operator is (minus) the Laplacian. If the plane is unbounded, the Green's function is $G = -(2\pi)^{-1} \log(|\mathbf{r} - \mathbf{r}'|)$. If $G \propto K_0(|\mathbf{r} - \mathbf{r}'|)$ (K_0 is the modified Bessel function), Eq. (1) then describes geostrophic flows [11], drift-electrostatic vortices [12], and electron vortices [13] in magnetized plasmas. Alfvén current-vortex filaments in a magnetized plasma show a mix of logarithmic and Bessel behavior [14]. The case $G \propto 1/|\mathbf{r} - \mathbf{r}'|$ corresponds to Hall vortices in a thin plasma slab [15]. Other examples can be found in [16]. Although we consider mainly the vortices in unbounded ideal fluids, our results are applicable to arbitrary systems of objects whose pairwise interaction energy in two dimensions is proportional to $G(\mathbf{r} - \mathbf{r}')$.

The statistics of uncorrelated vortices is usually studied from the point of view of the central limit theorem [17,18]. According to the standard formulation of this theorem [19] the probability density for the sum of N equally distributed independent random variables ξ_j ($j = 1, \ldots, N$) becomes Gaussian in the limit $N \to \infty$, provided that the second momentum (the variance) of the common distribution $f(\xi_j)$ exists, $\int \xi_j^2 f(\xi_j)\, d\xi_j < \infty$. The velocity of the ith vortex has the form $\mathbf{v}_i = \sum_j \mathbf{v}_{ij}$. Here, $\mathbf{v}_{ij}$ is the contribution from jth vortex, which can be treated as the random variable ξ_j. Since independently distributed point vortices can be arbitrarily closely placed near one another, the variance $\int \mathbf{v}_{ij}^2 f(\mathbf{v}_{ij})\, d\mathbf{v}_{ij}$ ($d\mathbf{v}_{ij}$ is the volume element in the velocity space) diverges and the central limit theorem, which would predict a Gaussian distribution for $\mathbf{v}_i$, need not be valid. In this paper we investigate vortex statistics according to

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the method introduced by Holtsmark [20]. This general statistical approach has been widely used in the theory of spectral line broadening (see [21] for a review) and has also been applied to describe the distribution of the gravitational force from stars that are randomly distributed in space [22]. As has been concluded in [23], and independently in [16], the Holtsmark method is also applicable to point-vortex systems.

We consider N point vortices randomly distributed inside the domain D of area S in a two-dimensional unbounded plane. The vortex number N is supposed to be large, but finite. The restriction to an unbounded region has been made mainly for convenience. In this case the Green's functions are simpler which enables us to get analytical estimates and to compare them with the results of numerical tests. The probability $f(\mathbf{r},\mathbf{v})d\mathbf{v}$ that the Nth vortex located at $\mathbf{r}$ has velocity between $\mathbf{v}$ and $\mathbf{v}+d\mathbf{v}$ is

$$f(\mathbf{r},\mathbf{v}) = \int \delta\Big(\mathbf{v} - \sum \kappa_j \mathbf{e}_z \times \nabla G(\mathbf{r}-\mathbf{r}_j)\Big)\sigma(\mathbf{r}_1,\kappa_1)\,d\mathbf{r}_1 d\kappa_1 \cdots \sigma(\mathbf{r}_{N-1},\kappa_{N-1})\,d\mathbf{r}_{N-1} d\kappa_{N-1}\,. \quad (2)$$

Here, δ is the delta function and $\sigma(\mathbf{r},\kappa)$ is the vortex distribution over space and strengths. We consider the asymptotic behavior of integral (2), for large (N,S) at a fixed ratio $n = N/S$. Expressing the delta function as a Fourier integral and assuming that all vortices have equal strength $\kappa_j = \kappa$, Eq. (2) can be reduced to [16,22]

$$f(\mathbf{r},\mathbf{v}) = \frac{1}{(2\pi)^2}\int \exp\left\{i\mathbf{k}\cdot\mathbf{v} - N\int\{1-\exp[i\kappa\mathbf{e}_z\cdot\mathbf{k}\times\nabla G(\mathbf{r}-\mathbf{r}')]\}\sigma(\mathbf{r}')\,d\mathbf{r}'\right\}d\mathbf{k}\,. \quad (3)$$

In what follows we consider the vortices to be uniformly distributed over the domain D, so that $\sigma(\mathbf{r}) = 1/S$ and the vortex density $n = N/S$ is constant.

We are interested in the difference $\mathbf{w} = \mathbf{v} - \mathbf{V}$ between the flow velocity (1) and its average value $\mathbf{V}$. The subtraction of the average velocity $\mathbf{V}$ is equivalent to introducing a uniform, neutralizing background. For $N \to \infty$ the distribution f does not depend on the vortex position $\mathbf{r}$ or on the shape of the domain. It is also clear that, due to symmetry, the function f depends only on $w \equiv |\mathbf{w}|$. First, we investigate hydrodynamic (Euler) point vortices with Green's function $G = -(2\pi)^{-1}\log(|\mathbf{r}-\mathbf{r}'|)$. We estimate the integral (3) in two different limits. The distribution function $f(w)$ at moderate velocities w is determined mainly by interactions between distant vortices. Considering a vortex placed at the center of a circular configuration and taking into account the leading term in N only in (3), we arrive at [16]

$$F(w) = \frac{w}{\bar{w}^2}\exp\left(-\frac{w^2}{2\bar{w}^2}\right), \qquad \bar{w} \simeq (n\kappa^2\Lambda/8\pi)^{1/2}, \qquad (w \le \bar{w})\,, \quad (4)$$

where $F(w) = 2\pi w f(w)$ and $\Lambda = \log N$. The high-velocity tail of the distribution function $f(w)$ is determined by the interaction between the vortex and its nearest neighbor. From the conditions $f(w)d\mathbf{w} = n d\mathbf{r}'$ and $w = |\kappa|/(2\pi|\mathbf{r}-\mathbf{r}'|)$, it follows that (see [16–18])

$$F(w) = (\kappa^2/2\pi)\,(n/w^3)\,, \qquad (w \gg \bar{w})\,. \quad (5)$$

Note that the function $F(w)$ given by Eq. (4) is normalized, $\int F(w) = 1$, without taking into account small corrections due to the high-velocity tail (5).

In order to verify Eqs. (4) and (5), we have carried out numerical simulations with $N = 1.6\times10^5$ identical point vortices. The vortices are placed randomly inside a circle of radius R. A circle with uniformly distributed vorticity density $\bar{\omega} = \kappa N/\pi R^2$ will rotate with constant angular velocity $\bar{\omega}/2$. It is convenient to introduce units in which this rotation velocity and the characteristic distance between vortices $\bar{d} = (S/\pi N)^{1/2}$ are equal to unity. This implies the choices $R = \sqrt{N} = 400$ and $\kappa = 2\pi$. Subtracting the solid rotation around the "vorticity center" of the configuration, we calculate the random velocity $\mathbf{w}$ and its distribution over the vortex array. This numerically obtained distribution function is shown in Fig. 1 by open circles. The solid line represents the analytical expression (4) with $\bar{w} = 2.7$. Figure 1 demonstrates that the bulk of the distribution function is well described by Eq. (4), although the value of $\bar{w}$ differs from the estimate $\bar{w} \simeq 2.5$ which follows from (4) for the actual values of N, R, and κ. Although the numerical distribution in Fig. 1 is nearly Gaussian, the non-Gaussian tail remains significant. This is clear from Fig. 2, where the data from Fig. 1 are shown by open circles together with the solid lines which represent the analytical expressions (4) and (5) with $\bar{w} = 2.7$. It is seen that the high-velocity tail is well approximated by Eq. (5).

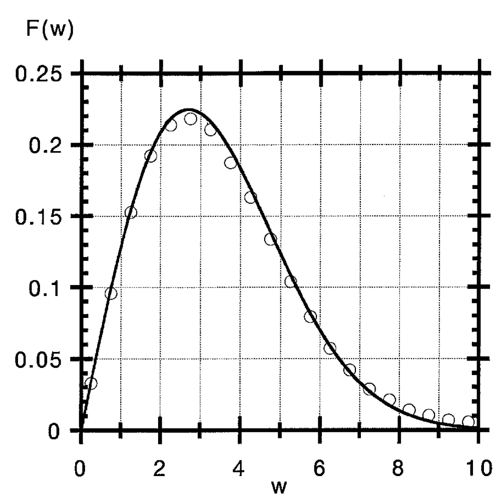

FIG. 1. Velocity distribution of identical, randomly placed Euler point vortices (open circles) along with Gaussian fit (4) (solid line).

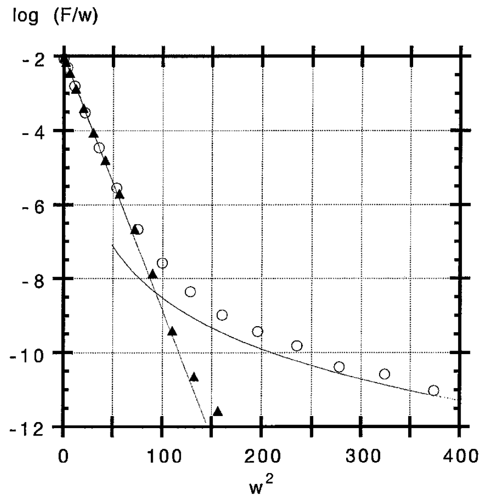

FIG. 2. Velocity distributions of regularly placed Euler vortices with random strength (solid triangles) and of randomly placed vortices with identical strength (open circles).

If the vortices do not have equal strength, then the integration in (2) should be performed both over vortex positions and strengths. One expects that, to a first approximation, (4) and (5) are still valid with κ being the averaged absolute value of the vortex strength, $\kappa \to \langle|\kappa|\rangle$. To check this, we have calculated the velocity distributions of randomly placed vortices with random strengths in the interval $[-4\pi, 4\pi]$. The resulting distribution function indeed agrees with (4) for $\bar{w} = 2.9$. In order to check that our results are not sensitive to the shape of the area that is occupied by the vortices, we have repeated the test placing vortices inside a square. Only minor changes in the bulk of the distribution function have been found. These numerical tests confirm the existence of a probability density $F(w)$ that converges to (4), in spite of the fact that integral (2) does not converge.

We have also calculated the velocity distribution of regularly placed vortices with uniformly distributed random strengths $\kappa_j \in [-4\pi, 4\pi]$. The vortices are placed on the nodes of the square grid, $x_l = l\sqrt{\pi}$, $y_m = m\sqrt{\pi}$, with $(l,m) = 0, \pm1, \pm2, \ldots,$ which provides approximately the same number of vortices inside the circle as in our first numerical test. The corresponding distribution, shown in Fig. 2 by solid triangles, is purely Gaussian.

In case of logarithmic interactions between vortices, the Holtsmark distribution is almost Gaussian. In particular, for this reason the Holtsmark origin of the velocity distribution has not been recognized in earlier numerical simulations [17,18]. This Gaussian character has the following explanation [16]. Consider the interaction of a vortex at the origin with vortices inside a region of fixed spatial angle and with characteristic scale L. This region contains $N' \simeq L^h$ vortices, where h is the spatial dimension. Statistical fluctuations of N' are $\sqrt{N'} \simeq L^{h/2}$. If ∇G scales as $1/|\mathbf{r} - \mathbf{r}'|^\alpha$, the velocity fluctuation of the vortex at the origin is $w \propto L^{h/2-\alpha}$. In the $2D$ case, $h = 2$ and $\alpha = 1$. This results in equal contributions to w from large and small scales and determines the Gaussian form of (4). The deviation of the Holtsmark from a Gaussian distribution can clearly be seen for nonlogarithmic Green's functions. We have carried out numerical tests with geostrophic vortices taking $G = (2\pi)^{-1}K_0(\gamma|\mathbf{r} - \mathbf{r}'|)$. In the limit $\gamma R \to 0$ the Euler case is reproduced, while for large values of γR the distributions are clearly non-Gaussian. As another example, we take Hall vortices with $G = 1/(2\pi|\mathbf{r} - \mathbf{r}'|)$ [15,16]. Equation (3) be integrated analytically [16],

$$F(w) = \frac{w/\sqrt{2}\,\bar{w}}{(1 + w^2/2\bar{w}^2)^{3/2}}, \qquad \bar{w} = n\langle|\kappa|\rangle/2\sqrt{2}. \quad (6)$$

The velocity distribution functions of randomly placed vortices with fixed strength and vortices with random strength which are regularly placed on the square grid are shown in Fig. 3 by open circles and solid triangles, respectively. They agree with the Holtsmark distribution (6) with $\bar{w} = 1/\sqrt{2} \simeq 0.7$ and the Gaussian distribution (4) with $\bar{w} = 0.65$, respectively. Note that Eq. (6) and Fig. 3 also describe the distribution of the gravitational field in disk galaxies, where the stars lie almost in a plane.

A more realistic model of turbulence introduces a minimal distance $d_{\min}$ between vortices [17]. The condition that the vorticity dissipation rate at $d = d_{\min}$ is comparable to the inverse characteristic time scale of the flow gives the estimate $d_{\min} \simeq \bar{d}/\sqrt{\mathrm{Re}}$, where $\mathrm{Re} = \bar{w}\bar{d}/\nu$ is the Reynold's number. Figure 3 shows velocity distributions of randomly placed vortices which have been generated under the restriction that the distance between them is larger than $d_{\min}$. The result proves that the Holtsmark distribution persists for $d_{\min}/\bar{d} < 0.1$.

For the logarithmic Green function the Holtsmark distribution manifests itself when the velocity difference of the flow induced by the vortices is measured. The velocity difference between two points $\Delta\mathbf{w}(\mathbf{r},\mathbf{d}) \equiv \mathbf{w}(\mathbf{r}+\mathbf{d}) - \mathbf{w}(\mathbf{r})$

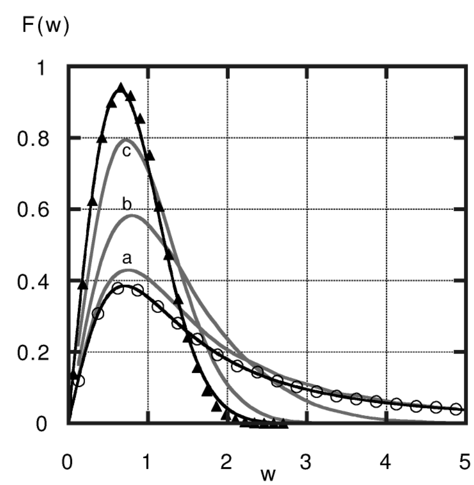

FIG. 3. Velocity distributions of regularly placed Hall vortices with $d_{\min} = \sqrt{\pi}\,\bar{d}$ (solid triangles) and randomly placed vortices with random strength (open circles). Gaussian (4) and Holtsmark (6) distributions (solid lines) hold, respectively. The intermediate solid curves show velocity distribution of vortices with $d_{\min}/\bar{d} = 0.4$ (a), 0.8 (b), 1.2 (c).

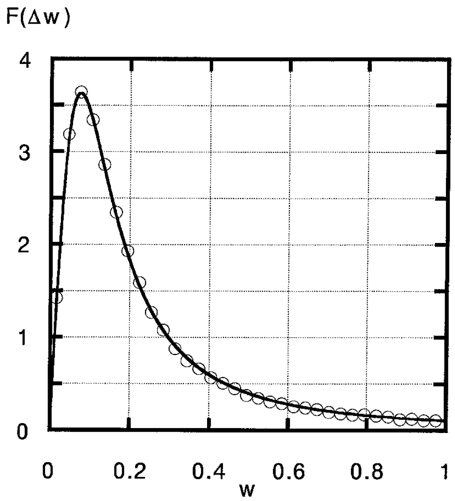

FIG. 4. The distribution of velocity differences (open circles) in a system of Euler vortices and the Holtsmark function (6) (solid line).

at distances $d \leq \bar{d}$ is approximately determined by the gradient of G which behaves as $1/r$. A straightforward calculation shows that the velocity difference distribution is given by Eq. (6) with $w \to \Delta w$ and $\bar{w} \to nd\langle|\kappa|\rangle/2\sqrt{2}$. In our last numerical test we have investigated the velocity difference field produced by randomly placed vortices with random strength. We have calculated the velocity differences between 10^5 pairs of points which are randomly sampled in space. The distance d_1 between these points is fixed, $d_1 = 0.1$. To avoid boundary effects, the points are picked near the center of the configuration. The numerical distribution function is shown in Fig. 4 by open circles. The solid line is the Holtsmark distribution (6) with $\bar{w} = 0.075$. This distribution is similar to the Lorentzian velocity difference distribution that has been observed experimentally [9].

In general, Holtsmark distributions are non-Gaussian when the central limit theorem is not applicable. We have demonstrated that non-Gaussian distributions occur even when the vortex velocity is limited from above and, hence, all momenta of the distribution $f(\mathbf{w}_j)$ exist. This happens because the central limit theorem imposes the additional condition [19] that an individual contribution does not dominate the total sum (1). This implies that the maximum possible velocity $w_{\max}$ due to the interaction between nearest neighbors, should not exceed $\bar{w}$. If there is no minimum separation between the vortices, this condition can never be reached, not even for the logarithmic Green's function. However, the relative number of particles in the high-velocity tail δN scales as $\delta N/N \propto 1/\log N$. This slow, non-normal convergence to a Gaussian distribution at $N \to \infty$ has also been observed in numerical experiments [17,18].

The point-vortex model of turbulence can also be applied to the 3D case when the Green's function in Eq. (1) is $G = -1/(4\pi|\mathbf{r} - \mathbf{r}'|)$ [23]. The contribution to random velocity from vortices at distances L scales as $w \propto L^{h/2-\alpha} \simeq L^{-1/2}$ for $h = 3, \alpha = 2$. The function $w(L)$ can decay even faster since the fractal dimension of the real turbulence is estimated both experimentally and theoretically (see [23] for references) to be $h \simeq 2.6 \pm 0.1$. Thus, due to the vanishing contribution from large distances, in the 3D case the point-vortex model predicts non-Gaussian distribution even for the velocity field. However, the deviation from a Gaussian is rather limited, except for the high-velocity tail [23]. The velocity difference distribution, on the other hand, will be strongly non-Gaussian.

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