Nonlinearly Saturated Dynamical State of a Three-Wave Mode-Coupled Dissipative System with Linear Instability

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Linearly unstable dissipative systems with quadratic nonlinearity occurring in plasma physics, optics, fluid mechanics, etc. are often modeled by a general set of three-wave mode-coupled ordinary differential equations for complex variables. Bounded attractors of the set approximate nonlinearly saturated turbulent states of real physical systems. Exact criteria for boundedness of the attractors are found. Fundamentally different kinds of asymptotic behavior of the wave triad are classified in the parameter space and quantitatively assessed.

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Dissipative nonlinear systems are widely present in many areas of science and technology [1]. Among them, rather common are systems with linear instabilities saturated by nonlinearity [2]. Understanding this process and finding quantitative characteristics of the saturated dynamical state represents a fundamental physical problem, which is often very complicated mathematically. However, the analysis may be simplified in cases when linearly unstable systems evolve to turbulent states with a restricted number of dominant wave perturbations [2,3]. Such nonlinearly saturated states can be modeled by attractors of truncated sets of coupled ordinary differential equations ("flows" [1]). For systems with a quadratic nonlinearity, three-wave models may reasonably describe key features of actual turbulence, especially if the systems are close to equilibrium. In the present Letter we study a three-wave model given by Eq. (1), which represents the most typical example of such a system. This or equivalent sets of equations arise in plasma physics [2], nonlinear optics [4], fluid mechanics [5], etc. In our case, this set of equations has been intended for modeling nonlinear saturation of dissipative low-frequency instabilities in the ionosphere [6,7]. The general results reported here are applicable to any problems reducible to this model.

Not far from the threshold of linear instability, when the energy level of saturated turbulence is anticipated to be reasonably low [7], dynamical behavior of three coupled harmonic waves in a quadratically nonlinear dissipative system is described by a set of three equations:

$$\frac{d\eta_j}{dt} + i\omega_j\eta_j - \gamma_j\eta_j = \rho_j\eta_{j-1}^*\eta_{j+1}^* \qquad (1)$$

with $j = 1, 2, 3, (\cdots)_{j+3} \equiv (\cdots)_j$. Complex variables η_j represent time-dependent amplitudes of spatial Fourier waves with given wave vectors $\mathbf{k}_j$ which are resonantly coupled, $\sum_{i=1}^{3}\mathbf{k}_i = 0$. Real linear wave frequencies $\omega_j(\mathbf{k}_j)$ and growth (damping) rates $\gamma_j(\mathbf{k}_j)$ are determined by specific dispersion relations. Mode-coupling coefficients ρ_j (also functions of $\mathbf{k}_i$) are generally complex. In many physically interesting cases, the complex arguments of ρ_j differ by 0, $\pm\pi$ [2–6]. In all these cases, via renormalization of variables, the mode-coupling coefficients ρ_j can be made real, which is implied below.

Asymptotic solutions at sufficiently large time t (i.e., attractors) depend strongly upon the parameters of the system and may also depend upon the initial conditions. In a linearly unstable situation when at least one γ_j in (1) is positive, there may be either bounded or unbounded asymptotic solutions for amplitudes of the interacting modes. Bounded solutions at $t \to \infty$ mean that there is nonlinear saturation of the instability via mode coupling. Although some properties of attractors of (1) have been studied before in particular cases [2–6], general issues crucial for the physics of nonlinear processes have not been addressed. Among the key questions are the following. What are the exact criteria for bounded asymptotic solutions? What are the dynamical properties of the corresponding attractors? What are the average energetic characteristics of the nonlinearly saturated state?

All these questions are addressed in the present Letter. For the general system (1) we have found exact analytical expressions for the full domain of parameters that provides saturation of the instability (saturation domain). We have also found an exact bifurcation surface separating two fundamentally different kinds of attractors. All analytical results have been checked numerically. The combination of analytical and numerical techniques has enabled us to classify in the parameter space different regimes of the attractor behavior and estimate the average energetic characteristics of the asymptotic solutions. Details of the general analysis, as well as application to a specific ionospheric problem [6,7], will be presented elsewhere.

Nonlinear saturation of the instability is possible only if all ρ_j are not of the same sign, otherwise explosive regimes eventually take place [2]. Without loss of generality, we may set $\rho_{1,2} < 0$, $\rho_3 > 0$. As shown below, bounded attractors may exist if only mode 3 is linearly unstable. Indeed, in a bounded asymptotic quasisteady state, for quantities averaged over a sufficiently long time (denoted by angular brackets), we have from (1)

$$\frac{\gamma_j}{\rho_j}\langle|\eta_j|^2\rangle = -\langle\mathrm{Re}(\eta_1\eta_2\eta_3)\rangle, \qquad (2)$$

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so that all three ratios (γ_j/ρ_j) must be of the same sign. Two linearly unstable modes usually cannot be stabilized by mode coupling with only one linearly stable mode, so that a necessary condition for a bounded attractor is given by $\gamma_3 > 0$, $\gamma_{1,2} < 0$. A full set of necessary and sufficient conditions is discussed below.

Equation (1) for complex variables η_j can be transformed to equations for real variables. By proper renormalization, the total number of parameters (nine) can be reduced to three. In order to keep the initial symmetry of (1), we do this in a form different from the conventional one [2]. Introducing a positive constant $W_\gamma = [\frac{1}{2}\sum_{j=1}^{3}(\gamma_j - \gamma_{j+1})^2]^{1/2}$, we renormalize the time, $\tau = W_\gamma t$, and define dimensionless parameters

$$\Delta = \frac{1}{W_\gamma}\sum_{j=1}^{3}\omega_j, \qquad \sigma = \frac{1}{W_\gamma}\sum_{j=1}^{3}\gamma_j, \tag{3a}$$

$$\Psi = \arccos\left(\frac{2\gamma_3 - \gamma_1 - \gamma_2}{2W_\gamma}\right). \tag{3b}$$

Parameter Δ characterizes the total frequency mismatch from the exact linear-frequency resonance $\sum_{j=1}^{3}\omega_j = 0$. Parameter σ characterizes the exponential rate of contraction or expansion of the elementary phase volume in the dissipative system [1]. Introducing dimensionless growth (damping) rates

$$g_j(\sigma,\Psi) = \frac{2\gamma_j}{W_\gamma} \equiv \frac{2}{3}(\sigma + 2\cos\Psi_j),$$

where $\Psi_j = \Psi - \frac{2\pi}{3}(j-3)$, and real variables v_j, X, Y,

$$v_j = \frac{\mathcal{P}}{2W_\gamma^2\rho_j}|\eta_j|^2, \qquad X + iY = \frac{\mathcal{P}}{W_\gamma^3}\eta_1\eta_2\eta_3, \tag{4}$$

where $\mathcal{P} = 8\rho_1\rho_2\rho_3 > 0$, $v_{1,2} \leq 0$, $v_3 \geq 0$, we obtain from (1) a five-dimensional flow

$$\frac{dv_j}{d\tau} = g_j(\sigma,\Psi)v_j + X,$$
$$\frac{dX}{d\tau} = \sigma X + \Delta Y + \frac{1}{2}(v_2v_3 + v_3v_1 + v_1v_2), \tag{5}$$
$$\frac{dY}{d\tau} = \sigma Y - \Delta X,$$

with the integral of motion $X^2 + Y^2 = v_1v_2v_3$; see (4).

There are two fundamentally different regimes of the asymptotic bounded solutions of Eqs. (5): (i) with stationary amplitudes of all modes (sink [1]) and (ii) with time-varying amplitudes: limit cycles or chaotic (strange) attractors. In the regime (i), the stationary amplitudes are given by $v_{j\mathrm{st}} = -X_{\mathrm{st}}/g_j$, $Y_{\mathrm{st}} = pX_{\mathrm{st}}$, $X_{\mathrm{st}} = -(1 + p^2)g_1g_2g_3$, where $p = \Delta/\sigma$. For the original variables $\eta_j(t)$, the corresponding solution is given by $\eta_j(t) = |\eta_j|_{\mathrm{st}}\exp[-i(\omega_j - p\gamma_j)t + i\varphi_{j0}]$, where

$$|\eta_j|_{\mathrm{st}}^2 = (1 + p^2)\frac{\gamma_{j+1}\gamma_{j-1}}{\rho_{j+1}\rho_{j-1}}, \tag{6}$$

and constant phase shifts φ_{j0} are locked by relation $\tan(\sum_{j=1}^{3}\varphi_{j0}) = p$. Analysis of this solution shows that it forms a stable attractor in a domain of parameters constrained by relations: $|\Psi| < \pi/3$,

$$\sigma_3(\Psi) < \sigma < \sigma_4(\Psi) \quad \text{at } 0 < |\Psi| < \frac{\pi}{6}, \tag{7a}$$

$$\sigma_3(\Psi) < \sigma < \sigma_{1,2}(\Psi) \quad \text{at } \frac{\pi}{6} < |\Psi| < \frac{\pi}{3} \tag{7b}$$

(see Fig. 1a), and $|\Delta| > \Delta_0(\sigma,\Psi)$ (see Fig. 1b),

$$\Delta_0 = |\sigma|\left[\frac{5\sigma^3 + 4\cos 3\Psi - 6\sigma\sqrt{4\sigma^2 + 4\sigma\cos 3\Psi + 1}}{3(\sigma^3 - 6\sigma - 4\cos 3\Psi)}\right]^{1/2}. \tag{8}$$

Here $\sigma_3(\Psi) = -2\cos\Psi$, $\sigma_{1,2}(\Psi) = -2\cos(|\Psi| - \frac{2\pi}{3})$,

$$\sigma_4(\Psi) = 2\sqrt{2}\cos\left[\frac{1}{3}\arccos\left(\frac{\cos 3\Psi}{\sqrt{2}}\right) - \frac{2\pi}{3}\right].$$

The surfaces $\sigma = \sigma_{1,2,3}(\Psi)$ correspond to $\gamma_{1,2,3} = 0$, respectively. The associated constraints express the fact that modes 1 and 2 are linearly stable and mode 3 is linearly unstable. Bifurcation surface $|\Delta| = \Delta_0(\sigma,\Psi)$, where $\Delta_0 \to \infty$ at $\sigma = \sigma_4(\Psi)$, gives the boundary of stability for the above stationary solution. Numerical calculations show that the stationary solution (6) under constraints (7) and (8) is the only attractor of the system, regardless of the initial conditions (i.e., one basin of attraction [1]).

For $|\Delta| < \Delta_0(\sigma,\Psi)$, no stable solutions with constant amplitudes exist, and all wave amplitudes are time varying. There are no exact analytic solutions for them. However, the exact domain of parameters where time-varying amplitudes are necessarily bounded can be found by using the following approach. For sufficiently large wave amplitudes, $|v_j| \gg 1$, most of the time the value of $|X|$ is large compared to $|g_jv_j|$ and $|Y|$. To the zeroth-order approximation, the small terms may be neglected and Eqs. (5) can be analytically solved [2]. The corresponding solution describes nonlinear oscillations of mode envelopes with amplitudes of oscillations inversely proportional to their period. These oscillations can be expressed in terms of the Weierstrass functions with two constant parameters. In the first-order approximation with finite values of g_j, these parameters are no longer constants, but adiabatically slowly vary with time. By using a two-timing procedure, one can obtain the corresponding evolutional equations and study their attractors. In the case under study, there can be only exponentially growing or damping mode amplitudes. The domain of parameters corresponding to exponentially damping oscillations is just the saturation domain for (1). A similar approach can be applied to finding saturation domains for other nonlinear systems. Note that the actual asymptotic regime in the saturation domain is given by sufficiently low-amplitude, long-periodic temporal variations that lie beyond the scope of the above adiabatic approximation.

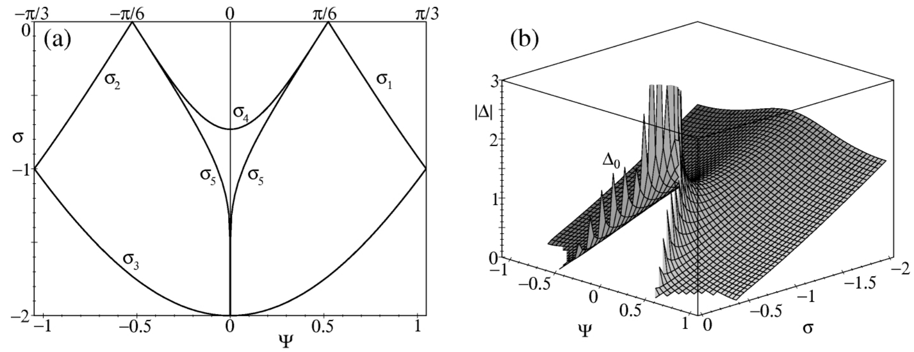

FIG. 1. Stability domain in the space of reduced parameters; see (3). (a) Constraints in the σ-Ψ plane; see (7) and (9). (b) Bifurcation surface between the sink and time-varying solutions; see (8).

Implementing the above procedure, we obtain that at $|\Delta| < \Delta_0(\sigma, \Psi)$ the asymptotic time-varying solutions are bounded under constraints (7), where $\sigma_4(\Psi)$ is replaced by another, parametrically defined function $\sigma_5(\Psi)$,

$$\sigma_5(\Psi) = 2\left[\sqrt{3}\sin\left(|\Psi| + \frac{2\pi}{3}\right)\mathcal{R}(k) - \cos\Psi\right],$$
$$\Psi = \pm\arctan\left[\frac{\sqrt{3}(1 - k^2)[1 - \mathcal{R}(k)]}{(1 + k^2)\mathcal{R}(k) + k^2 - 1}\right]. \quad (9)$$

Here $\mathcal{R}(k) = \mathbf{E}(k)/\mathbf{K}(k)$, where $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are the complete elliptic integrals of the 1st and 2nd kind, respectively. The boundary $\sigma = \sigma_5(\Psi)$ (Fig. 1a) is composed of two smooth symmetric curves which exponentially closely converge to axis $\Psi = 0$ corresponding to the degenerate case of $\gamma_1 = \gamma_2$ [3].

Numerical calculations for asymptotic temporal behavior of wave amplitudes at $|\Delta| < \Delta_0(\sigma, \Psi)$ have fully confirmed all these results. In particular, in the area between the two converging parts of the boundary $\sigma = \sigma_5(\Psi)$, the wave amplitudes nonlinearly oscillate and exponentially grow with no bound [8], following the adiabatic scenario described above; see Fig. 2a. The exponential growth rate tends to zero on approaching the boundary. Within the saturation areas we have bounded only asymptotic solutions whose character and average amplitudes strongly

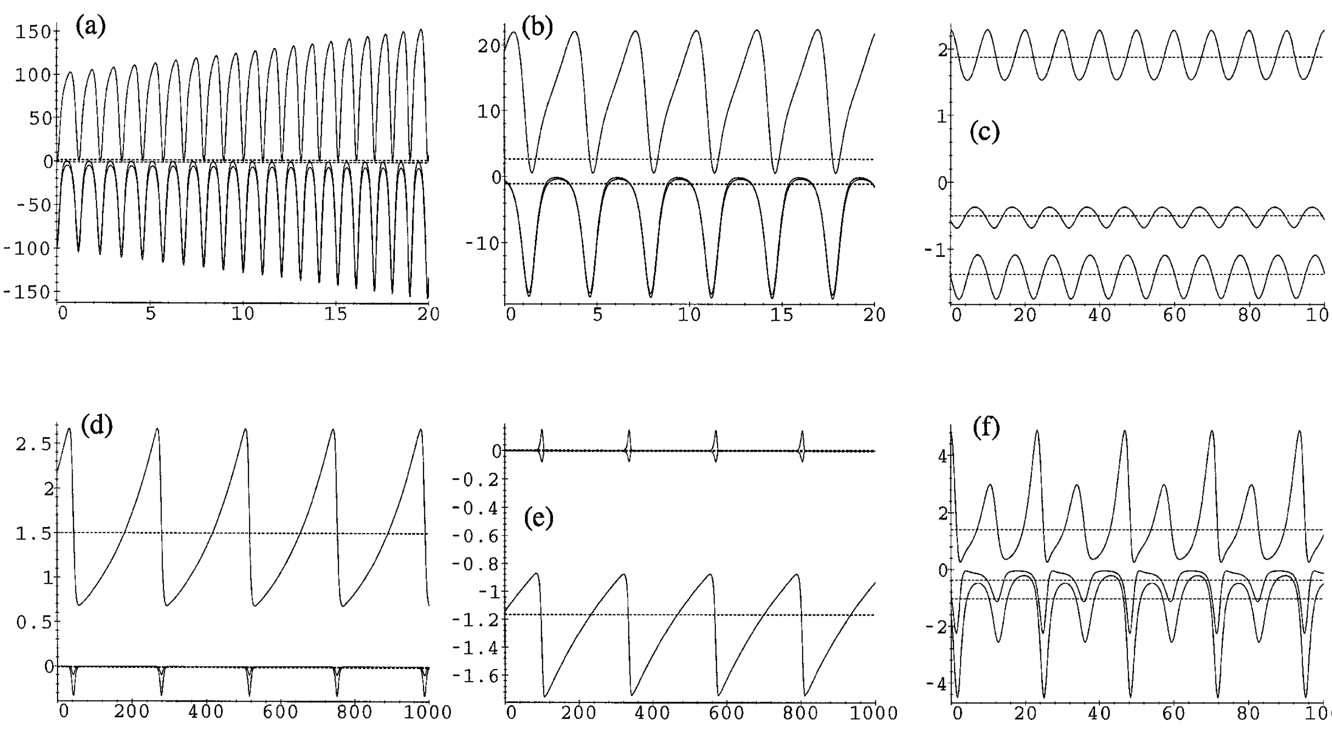

FIG. 2. Examples of oscillating asymptotic solutions (solid curves) with the corresponding stationary values (dashed lines); see (6). Vertical: dimensionless amplitudes v_j, ordered according to $v_1 \le v_2 \le 0 \le v_3$; horizontal: dimensionless time τ. (a) $\Psi = 0.083$, $\sigma = \sigma_5(\Psi) + 0.02 \approx -0.9354$, $\Delta = 0$; (b) $\Psi = 0.0409$, $\sigma = \sigma_5(\Psi) - 0.01 \approx -1.111$, $\Delta = 0.5\Delta_0(\sigma, \Psi) \approx 0.6445$; (c) $\Psi = \pi/6$, $\sigma = -1$, $\Delta = \Delta_0(\sigma, \Psi) - 0.01 \approx 0.739$; (d) $\Psi = 0.75$, $\sigma = \sigma_3(\Psi) + 0.01 \approx -1.4534$, $\Delta = 0$; (e) $\Psi = 0.75$, $\sigma = \sigma_1(\Psi) + 0.005 \approx -0.356$, $\Delta = 0$; (f) $\Psi = \pi/6$, $\sigma = -1$, $\Delta = 0.4$.

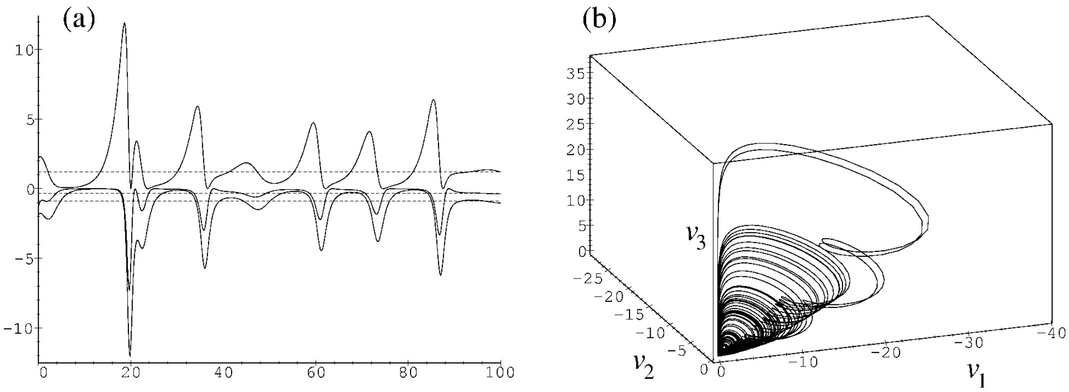

FIG. 3. (a) Example of chaotic asymptotic solution, arranged similar to Fig. 2 with (b) the corresponding strange attractor in the phase space: $\Psi = \pi/6$, $\sigma = -1$, and $\Delta = 0$.

vary depending upon the position in the three-dimensional parameter space, but with no observed dependence on the initial conditions.

Near all boundaries, including the bifurcation surface $|\Delta| = \Delta_0(\sigma, \Psi)$, nonlinearly saturated oscillations have simple periodic character. Near the boundaries $\sigma = \sigma_5(\Psi)$ (Fig. 2b) and $|\Delta| = \Delta_0(\sigma, \Psi)$ (Fig. 2c), such oscillations occupy rather thick continuous volumes in the parameter space. By contrast, periodic solutions near the boundaries $\sigma = \sigma_{1,2}(\Psi)$ are localized in very thin layers. Saturated temporal variations have here the character of long-periodic sawtooth oscillations of the largest wave amplitude with short-time bursts of two smaller amplitudes; see Figs. 2d and 2e. The largest amplitude corresponds to the mode whose growth or damping rate is close to zero; see Eq. (2). The amplitude of sawtooth oscillations is inversely proportional to their period; the latter increases as the corresponding boundary is approached. On passing to the deep inner part of the saturation domain, temporal behavior of wave amplitudes, through more complex limit cycles (Fig. 2f), transforms to chaotic (Fig. 3). Subdomains of parameters corresponding to different kinds of attractors may have a complicated structure [9]. This issue and fractal properties of strange attractors [1,3] need additional study.

Energetic characteristics of the nonlinearly saturated state can be estimated via average oscillation amplitudes $\langle|\eta_j|^2\rangle$ [see Eq. (2)] by comparing them with the stationary amplitudes $|\eta_j|^2_{\rm st}$ given by Eq. (6). Computations show that near all boundaries of the saturation domain for time-varying solutions, excluding $\sigma = \sigma_5(\Psi)$, we have $\langle|\eta_j|^2\rangle \approx |\eta_j|^2_{\rm st}$; very close to $\sigma = \sigma_5(\Psi)$, $\langle|\eta_j|^2\rangle \gg |\eta_j|^2_{\rm st}$. In the interior of the domain, we have $\langle|\eta_j|^2\rangle \simeq |\eta_j|^2_{\rm st}$.

Analysis of applicability of the truncated three-wave model to real systems and self-consistent determination of the preferred wave triad should be done with application to specific physical problems. Such analysis for ionospheric instabilities [6] shows that a background of initially chaotically excited waves may really evolve to a dynamical structure resembling a wave triad with preferred wave vectors $\mathbf{k}_j$ [7]. There is a tendency for $\mathbf{k}_j$ to be located at a margin of linear stability for the most intense wave, with the maximum permissible value of the corresponding $|\eta_j|_{\rm st}$ (often at a sink regime, $|\Delta| > \Delta_0$). Some of these features may hold in other problems.

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