Topological Evolution of Dynamical Networks: Global Criticality from Local Dynamics

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We evolve network topology of an asymmetrically connected threshold network by a simple local rewiring rule: quiet nodes grow links, active nodes lose links. This leads to convergence of the average connectivity of the network towards the critical value $K_c = 2$ in the limit of large system size N. How this principle could generate self-organization in natural complex systems is discussed for two examples: neural networks and regulatory networks in the genome.

Networks of many interacting units occur in diverse areas as, for example, gene regulation, neural networks, food webs in ecology, species relationships in biological evolution, economic interactions, and the organization of the internet. For studying statistical mechanics properties of such complex systems, discrete dynamical networks provide a simple testbed for effects of globally interacting information transfer in network structures.

One example is the threshold network with sparse asymmetric connections. Networks of this kind were first studied as diluted, nonsymmetric spin glasses [1] and diluted, asymmetric neural networks [2,3]. For the study of topological questions in networks, a version with discrete connections $c_{ij} = \pm 1$ is convenient and will be considered here. It is a subset of Boolean networks [4,5] with similar dynamical properties. Random realizations of these networks exhibit complex non-Hamiltonian dynamics including transients and limit cycles [6,7]. In particular, a phase transition is observed at a critical average connectivity K_c with lengths of transients and attractors (limit cycles) diverging exponentially with system size for an average connectivity larger than K_c. A theoretical analysis is limited by the non-Hamiltonian character of the asymmetric interactions, such that standard tools of statistical mechanics do not apply [1]. However, combinatorial as well as numerical methods provide a quite detailed picture about their dynamical properties and correspondence with Boolean networks [6–15].

While basic dynamical properties of interaction networks with fixed architecture have been studied with such models, the origin of specific structural properties of networks in natural systems is often unknown. For example, the observed average connectivity in a nervous structure or in a biological genome is hard to explain in a framework of networks with a static architecture. For the case of regulation networks in the genome, Kauffman postulated that gene regulatory networks may exhibit properties of dynamical networks near criticality [4,16]. However, this postulate does not provide a mechanism able to generate an average connectivity near the critical point. An interesting question is whether connectivity may be driven towards a critical point by some dynamical mechanism. In the following, we will sketch such an approach in a setting of an explicit evolution of the connectivity of networks.

Network models of evolving topology, in general, have been studied with respect to critical properties earlier in other areas, e.g., in models of macroevolution [17]. Network evolution with a focus on gene regulation has been studied first for Boolean networks in [18] observing self-organization in network evolution, and for threshold networks in [19]. Combining the evolution of Boolean networks with game theoretical interactions is used to model networks in economy [20].

In a recent paper, Christensen *et al.* [21] introduce a static network with evolving topology of undirected links that explicitly evolves towards a critical connectivity in the largest cluster of the network. In particular, they observe for a neighborhood-oriented rewiring rule that the connectivity of the largest cluster evolves towards the critical $K_c = 2$ of a marginally connected network. Motivated by this work, we here consider the topological evolution of threshold networks with asymmetric links to study how local rules may affect global connectivity of a network, including the entire set of clusters of the network.

In the remainder of this Letter we define a threshold network model with a local, topology-evolving rule. Then numerical results are presented that indicate an evolution of topology towards a critical connectivity in the limit of large system size. Finally, we discuss these results with respect to other mechanisms of self-organization and point to possible links with interaction networks in natural systems.

Let us consider a network of N randomly interconnected binary elements with states $\sigma_i = \pm 1$. For each site i, its state at time $t + 1$ is a function of the inputs it receives from other elements at time t:

$$\sigma_i(t + 1) = \text{sgn}[f_i(t)], \qquad (1)$$

with

$$f_i(t) = \sum_{j=1}^{N} c_{ij}\sigma_j(t) + h. \qquad (2)$$

The interaction weights c_{ij} take discrete values $c_{ij} = \pm 1$, with $c_{ij} = 0$ if site i does not receive any input from

element j. In the following, the threshold parameter h is set to zero. The dynamics of the network states is generated by iterating this rule starting from a random initial condition, eventually reaching a periodic attractor (limit cycle or fixed point).

Then we apply the following local rewiring rule to a randomly selected node i of the network.

If node i does not change its state during the attractor, it receives a new nonzero link c_{ij} from a random node j. If it changes its state at least once during the attractor, it loses one of its nonzero links c_{ij}.

Iterating this process leads to a self-organization of the average connectivity of the network.

To be more specific, let us now describe one of several possible realizations of such an algorithm in detail. We define the average activity $A(i)$ of a site i:

$$A(i) = \frac{1}{T_2 - T_1} \sum_{t=T_1}^{T_2} \sigma_i(t), \qquad (3)$$

where the sum is taken over the dynamical attractor of the network defined by T_1 and T_2. For practical purposes, if the attractor is not reached after T_{max} updates, $A(i)$ is measured over the last $T_{max}/2$ updates. This avoids exponential slowing down by long attractor periods for an average connectivity $K > 2$. The algorithm is then defined as follows: (i) Choose a random network with an average connectivity K_{ini}. (ii) Choose a random initial state vector $\vec{\sigma}(0) = (\sigma_1(0), \ldots, \sigma_N(0))$. (iii) Calculate the new system states $\vec{\sigma}(t), t = 1, \ldots, T$ according to Eq. (2), using a parallel update of the N sites. (iv) Once a previous state reappears (a dynamical attractor is reached), or otherwise after T_{max} updates the simulation is stopped. Then change the topology of the network according to the following local rewiring rule. (v) A site i is chosen at random and its average activity $A(i)$ is determined. (vi) If $|A(i)| = 1$, i receives a new link c_{ij} from a site j selected at random, choosing $c_{ij} = +1$ or -1 with equal probability. If $|A(i)| < 1$, one of the existing nonzero links of site i is set to zero. (vii) Finally, one nonzero entry of the connectivity matrix is selected at random and its sign reversed. (vii) Go to step (ii) and iterate.

The fluctuations introduced in step (vii) as random spin reversals are motivated by structurally neutral noise often observed in natural systems. Omitting this step does not change the basic behavior of the algorithm; however, the distribution of the number of inputs per node then evolves away from a Poissonian, thereby increasing the fraction of nodes with many inputs. The resulting dynamics differs only from the original algorithm in a slightly larger connectivity K_{ev} of the evolved networks. This effect vanishes $\sim 1/N$ with increasing system size.

The typical picture arising from the model as defined above is shown in Fig. 1 for a system of size $N = 1024$. Independent of the initial connectivity, the system evolves towards a statistically stationary state with an average con-
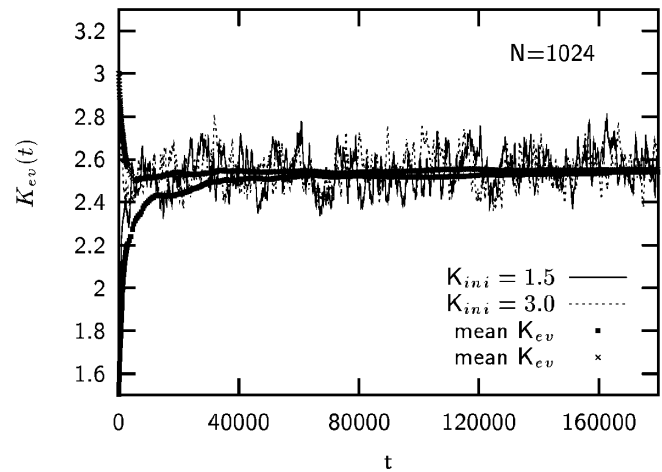

FIG. 1. Evolution of the average connectivity of threshold networks rewired according to the rules described in the text, for $N = 1024$ and two different initial connectivities ($K_{ini} = 1.5$ and $K_{ini} = 3.0$). Independent of the initial conditions chosen at random, the networks evolve to an average connectivity $K_{ev} = 2.55 \pm 0.04$. The plot shows the time series and the corresponding cumulative means for K_{ev}. The evolutionary time t is discrete, each time step representing a dynamical run on the evolved topology. Individual runs were limited to $T_{max} = 1000$ iterations.

nectivity $K_{ev}(N = 1024) = 2.55 \pm 0.04$. With varying system size we find that with increasing N the average connectivity converges towards K_c (which, for threshold $h = 0$ as considered here, is found at $K_c = 2$), (see Fig. 2). One observes the scaling relationship,
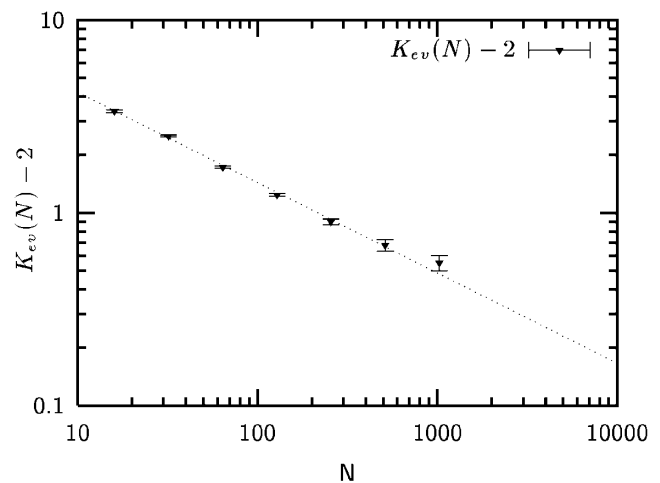
$$K_{ev}(N) - 2 = cN^{-\delta}, \qquad (4)$$

FIG. 2. The average connectivity of the evolved networks converges towards K_c with a scaling law $\sim N^{-\delta}$, $\delta = 0.47 \pm 0.01$. For systems with $N \leq 256$, the average was taken over 4×10^6 time steps, for $N = 512$ and $N = 1024$ over 5×10^5 and 2.5×10^5 time steps, respectively. Finite size effects from $T_{max} = 1000$ may overestimate K_{ev} for the largest network shown here.

with $c = 12.4 \pm 0.5$ and $\delta = 0.47 \pm 0.01$. Thus, in the large system size limit $N \to \infty$ the networks evolve towards the critical connectivity $K_c = 2$.

The self-organization towards criticality observed in this model is different from currently known mechanisms exhibiting the amazingly general phenomenon of self-organized criticality (SOC) [17,22,23]. Our model introduces a (new, and interestingly different) type of mechanism by which a system self-organizes towards criticality, here $K \to K_c$. In particular, it exhibits considerable robustness against noise in the system. This class of mechanisms lifts the notions of SOC to a new level. The main mechanism here is based on a topological phase transition in dynamical networks. To see this, consider the statistical properties of the average activity $A(i)$ of a site i for a random network. It is closely related to the frozen component $C(K)$ of the network, defined as the fraction of nodes that do not change their state along the attractor. The average activity $A(i)$ of a frozen site i thus obeys $|A(i)| = 1$. In the limit of large N, $C(K)$ undergoes a transition at K_c vanishing for larger K. With respect to the average activity of a node, $C(K)$ equals the probability that a random site i in the network has $|A(i)| = 1$. Note that this is the quantity which is checked stochastically by the local update rule in the above algorithm. The frozen component $C(K, N)$ is shown for random networks of two different system sizes N in Fig. 3. One finds that $C(K, N)$ can be approximated by

$$C(K, N) = \frac{1}{2} \{1 + \tanh[-\alpha(N)(K - K_0(N))]\}. \quad (5)$$

This describes the transition of $C(K, N)$ at an average connectivity $K_0(N)$ which depends only on the system size N. One finds for the finite size scaling of $K_0(N)$ that

$$K_0(N) - 2 = aN^{-\beta}, \quad (6)$$

with $a = 3.30 \pm 0.17$ and $\beta = 0.34 \pm 0.01$ (see Fig. 4), whereas the parameter α scales with system size as

$$\alpha(N) = bN^{\gamma}, \quad (7)$$

with $b = 0.14 \pm 0.016$ and $\gamma = 0.41 \pm 0.01$. Thus, we see that in the thermodynamic limit $N \to \infty$ the transition from the frozen to the chaotic phase becomes a sharp step function at $K_0(\infty) = K_c$. These considerations apply well to the evolving networks in the rewiring algorithm.

In addition to the rewiring algorithm as described in this Letter, we tested a number of different versions of the model. Including the transient in the measurement of the average activity $A(i)$ results in a similar overall behavior (where we allowed a few time steps for the transient to decouple from initial conditions). Another version succeeds using the correlation between two sites instead of $A(i)$ as a mutation criterion (this rule could be called "anti-Hebbian" as in the context of neural network learning). In addition, this version was further changed allowing different locations of mutated links, both between the tested sites or just at one of the nodes. Some of these versions will be discussed in detail in a separate article [24]. All these different realizations exhibit the same basic behavior as found for the model above. Thus, the mechanism proposed in this Letter exhibits considerable robustness.
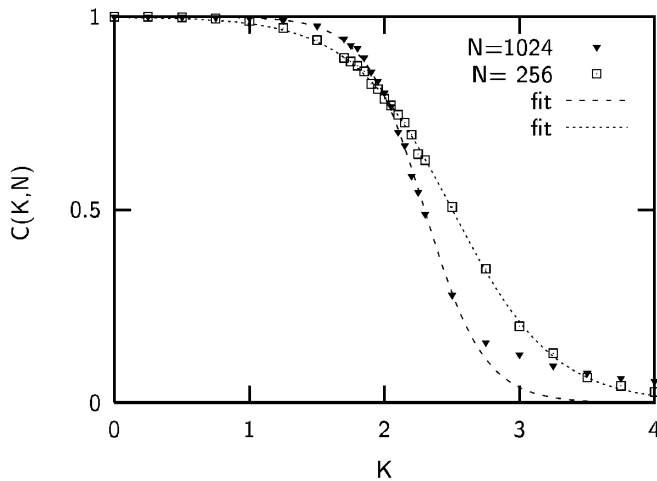

FIG. 3. The frozen component $C(K, N)$ of random threshold networks, as a function of the networks' average connectivities K. For both system sizes shown here ($N = 256$ and $N = 1024$), the data were measured along the dynamical attractor reached by the system, averaged over 1000 random topologies for each value of K. One observes a transition around a value $K = K_0$ approaching $K_c = 2$ for large N. A sigmoid function fit is also shown. To avoid trapping in exponential divergence of attractor periods for $K > 2$, the simulations have been limited to $T_{max} = 10\,000$. The mismatch of data and fit for $N = 1024$, $K \geq 2.75$ is due to this numerical limitation.
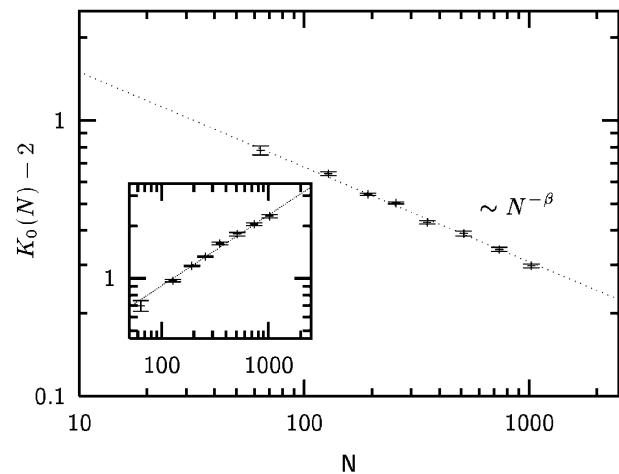

FIG. 4. The finite size scaling of the transition value K_0, obtained from sigmoidal fits as shown in Fig. 3. K_0 approaches $K_c = 2$ with a scaling law $\sim N^{-\beta}$, $\beta = 0.34 \pm 0.01$. The inset shows the scaling behavior of the parameter $\alpha(N)$; one finds $\alpha(N) \sim N^{\gamma}$, $\gamma = 0.41 \pm 0.014$.

An interesting question is whether a comparable mechanism may occur in natural complex systems, in particular, whether it could lead to observable consequences that cannot be explained otherwise.

One example where such mechanisms could occur is the regulation of connectivity density in neural systems. Activity-dependent attachment of synapses to a neuron is well known experimentally, for example, in the form of the gating of synaptic changes by activity correlation between neurons [25]. Such local attachment rules could provide a sufficient basis for a collective organization to occur as described in this Letter. For symmetric neural networks similar rules have been discussed, e.g., in the context of "Hebbian unlearning" suppressing spurious memories [26]. In the asymmetric networks studied here, however, such rules appear to generate a completely new form of self-organization dynamics. As a consequence, an emerging average connectivity K_{ev} could be stabilized to a specific value mostly determined by local properties of the dynamical elements of the system. It would be interesting to discuss whether synaptic density in biological systems could be regulated by such mechanisms.

Another biological observable of interest is the connectivity of gene-gene interactions in the expression of the genome as first studied by Kauffman [4]. Whether this observable results from any such mechanism clearly is an open question. However, one may discuss whether biological evolution exerts selection pressure on the single gene level, that results in a selection rule similar to our algorithm; for example, for a frozen regulation gene which is practically nonfunctional to obtain a new function (obtain a new link), as well as for a quite active gene to reduce functionality (remove a link). First experimental estimates for the global observable of genome connectivity are available for *E. coli* with a value in the range $2-3$ [27]. While it is clearly too early to speculate about the mechanisms of global genome organization, it is interesting to note that the robust self-organizing algorithm presented here provides a mechanism that, in principle, predicts a value in this range.

In summary, we study topological evolution of asymmetric dynamical networks on the basis of a local rewiring rule. We observe a network evolution with the average connectivity K of the network evolving towards the critical connectivity K_c without tuning. In the limit of large system size N, this limit becomes accurate. It is well conceivable that this form of global evolution of a network structure towards criticality might be found in natural complex systems.

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