Condensation of "Composite Bosons" in a Rotating BEC

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We provide evidence for several novel phases in the dilute limit of rotating Bose-Einstein condensates. By exact calculation of wave functions and energies for small numbers of particles, we show that the states near integer angular momentum per particle are best considered condensates of composite entities, involving vortices and atoms. We are led to this result by explicit comparison with a description purely in terms of vortices. Several parallels with the fractional quantum Hall effect emerge, including the presence of the Pfaffian state.

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In rotating superfluid ^{4}He a vortex lattice forms which, on scales large compared to the vortex lattice parameter, has a velocity field indistinguishable from a rigid body corotating with the container. The vortices only perturb the fluid density significantly over a region of order the coherence length, ξ, around the core of each vortex (of the order of an angstrom). Hence the arrangement of the vortex lattice is governed by minimizing the *kinetic* energy of the fluid in the rotating frame. One may say that the potential energy is "quenched" by the incompressibility of the fluid.

In the Bose condensed alkali gases, although so far it has proved difficult experimentally to investigate the rotational properties of the condensates, there has been a vigorous theoretical debate [1–3] about the stability (or otherwise) of vortices in the condensates. At a mean field level (appropriate for moderate density), the inhomogeneity of the condensate density and the existence of surface waves due to the harmonic well makes the description difficult. Nevertheless the *interparticle* potential energy is still largely unaffected by the presence of vortices in the limit where the coherence length is small compared to the extent of the condensate: it is the kinetic energy (and the single-particle trap potential) which determines the vortex positions.

In this Letter we show that when the coherence length is comparable to the extent of the condensate completely new effects occur. This is due to the *kinetic* (and single-particle trap) energy being quenched, by a combination of spherical symmetry and the special properties of the harmonic well. Hence the ground state in the rotating frame is determined by the interparticle interactions alone, reminiscent of the fractional quantum Hall effect. Indeed, we find stable states that are related to those found in the Hall effect (albeit in the less familiar regime of filling fraction, $\nu \gtrsim 1$). These include "condensates" of composite bosons of the atoms attached to an integral number of quanta of angular momenta, as well as the Laughlin and Pfaffian [4] states.

In a rotating reference frame, the standard Hamiltonian for N weakly interacting atoms in a trap is [5]

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N}\left(-\nabla_i^2 + r_i^2 + \eta\sum_{j=1,\neq i}^{N}\delta(\mathbf{r}_i - \mathbf{r}_j) - 2\boldsymbol{\omega}\cdot\mathbf{L}_i\right), \tag{1}$$

where we have used the trap energy, $\hbar\sqrt{K/m} = \hbar\omega_0$, as the unit of energy and the extent, $(\hbar^2/MK)^{1/4}$, of the harmonic oscillator ground state as the unit of length. (Here M is the mass of an atom and K the spring constant of the harmonic trap.) The coupling constant is defined as $\eta = 4\pi\bar{n}a(\hbar^2/MK)^{-1/2}$ where $\bar{n}$ is the average atomic density and a the scattering length. The angular velocity of the trap, ω, is measured in units of the trap frequency.

In the dilute limit $\eta \ll 1$, which implies in existing experimental traps that the number of atoms, N, would be $10 \lesssim N \lesssim 1000$. Then the average coherence (or healing) length is $\xi \sim 1/\sqrt{\bar{n}a} \to \infty$. It has been shown previously [6] that in this limit the problem becomes two dimensional and the Hilbert space may be truncated to the "lowest Landau level" states [7], $\psi_m(z) \propto z^m e^{-|z|^2/2}$, where $m \geq 0$ and $z = x + iy$ in the plane normal to $\boldsymbol{\omega}$. Indeed, at $\omega = 1$ the problem is identical to the quantum Hall problem, with $\boldsymbol{\omega}$ replacing the magnetic field.

We have determined the exact ground state, its energy $E_0(\omega)$, and excitation gap Δ for the Hamiltonian Eq. (1) using MATHEMATICA for $N \leq 8$ and $\omega \leq 1$. In addition, we have determined numerically the lowest eigenvalues for $N \leq 10$ as a function of ω. $L_0(\omega)$, the angular momentum of the ground state, is plotted in Fig. 1 for $N = 6$ with $\eta = 1/N$. Angular momentum remains a good quantum number as we have made no symmetry breaking ansatz.

A corresponding plot for ^{4}He in a rotating container would show jumps in the *expectation value* of $L_0(\omega)$ as successive vortices enter the system. The inhomogeneous density of the condensate in a trap leads to more complex, but similar, behavior in a mean field treatment [8] (appropriate in the high density limit). There are a number of important features in Fig. 1, which are common to all values of N which we have studied.

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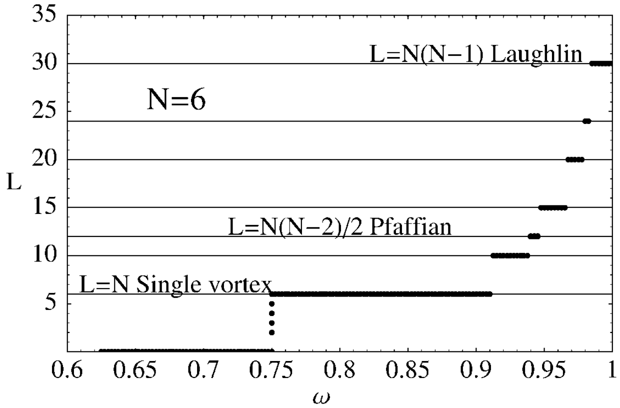

FIG. 1. Stable states for $N = 6$ in the rotating frame with $\eta = 1/N$.

First, at $L = N$ there is a state which corresponds [6] to one vortex,

$$\psi^{1\mathrm{v}}(\{z_i\}) = \prod_{i=1}^{N}(z_i - z_c)e^{-|\mathbf{z}|^2/2}, \tag{2}$$

where $z_c = (\sum_{i=1}^{N} z_i)/N$ is the center of mass coordinate and $|\mathbf{z}|^2 = \sum_{i=1}^{N}|z_i|^2$. From this point we will omit normalization factors and the ubiquitous $e^{-|\mathbf{z}|^2/2}$. This state has an interparticle interaction energy $E = \eta N(N-2)/4$ and becomes stable at $\omega_1 = (1 - N\eta/4)$. (In addition, at ω_1 all $N \geq L > 1$ states are metastable [9].)

Almost all of the other stable states can be labeled by $L = n(N - m)$ where n and m are non-negative integers (this includes the Laughlin state $n = N$, $m = 1$). These values are close to "n-vortex states" ($L = nN$), a possibility we will return to. However, the actual wave functions for these states most closely resemble some [10] used in the theory of composite fermions [11] in the Hall effect.

We define

$$Q_n(z_i) = \frac{\partial^{(N-1-n)}}{\partial z_i^{(N-1-n)}} \prod_{j=1, j\neq i}^{N}(z_i - z_j).$$

[Note $\psi^{\mathrm{Lau}} = \prod_{i=1}^{N} Q_{N-1}(z_i)$ and $\psi^{1\mathrm{v}} = \prod_{i=1}^{N} Q_1(z_i)$.] Then the states of high overlap with the true states at $L = n(N - m)$ may be written as

$$\psi_{n,m}(\{z_i\}) = \sum_{j_1<j_2<\cdots<j_{(N-m)}}^{N} Q_n(z_{j_1})Q_n(z_{j_2})\ldots Q_n(z_{j_{(N-m)}}).$$

Table I shows the overlaps of $\psi_{n,m}$ with the true ground states for those L. Their construction ensures that angular momentum is used economically to lower the energy: any given particle pair, i and j, will at most be associated with two factors of $(z_i - z_j)$.

The interpretation of the states at $L = n(N - m)$ is that a particle in association with n quanta of angular momentum is a particularly stable entity in the vicinity of $L = nN$. As L is reduced, $N - m$ particles remain with all n quanta and m have all the angular momentum removed. This is our main result, which occurs at small angular momenta. This is reminiscent of the "bound state" composite fermions of electrons and vortices [12]. We will return to this point.

We will now attempt to reconcile the composite boson states to the vortex states found in the nonlinear Schrödinger equation [3]. The following argument indicates a connection. Consider incompressible irrotational fluid ("helium") in a two-dimensional circular container, of radius R, with n point vortices at radial coordinates r_α. There the angular momentum of the fluid is [13]

$$L(\{r_\alpha\}) = N\left(n - \sum_{\alpha=1}^{n}(r_\alpha/R)^2\right); \tag{3}$$

i.e., the angular momentum is reduced from $L = nN$ by the vortices being off center.

To test this notion, we first localize the vortices (resulting in a nonrotationally invariant state) by superposing states with different L. Using $L = 10$ and $L = 8$ for $N = 5$ (the Pfaffian state rules out $N = 6$) yields the contour plot of probability density, Fig. 2. The two dimples might be interpreted as two off-center vortices (hence the angular momentum is lower than $L = 2N$). The figure is reminiscent of the figures in [8], although the changes in density are rather small by comparison. Note, however, the superposition is certain to create features periodic with $\cos 2\theta$, where θ is the polar angle.

To quantify these ideas, we introduce the lowest Landau level vortex factors, $\prod_{i=1}^{N}(\zeta_\alpha - z_i)$, with complex vortex

TABLE I. Stable states for $N \leq 8$: the upper number is their angular momentum and the lower is their overlap with the Q wave functions. ♠ indicates that the wave function can also be written (or derived from) a Pfaffian state.

	$2N-4$	$2N-2$	$2N$	$3N-6$	$3N-3$	$3N$	$4N-8$	$4N-4$	$4N$	$5N-10$	$5N-5$	$5N$	$6N-6$	$6N$
5		**8** ♠	**10**		**12**	**15**			**20**					
		0.98^2	0.87^2		0.99^2	0.96^2			1					
6		**10** ♠	**12** ♠	**12** ♠	**15**			**20**	**24**			**30**		
		0.86^2	0.69^2	0.79^2	0.95^2			0.99^2	0.94^2			1		
7		**12**		**15**	**18** ♠			**24**			**30**	**35**		**42**
		0.83^2		0.49^2	0.85^2									1
8	**12**	**14**		**18**		**24** ♠	**24** ♠		**32**	**30**	**35**		**42**	···
	0.66^2													

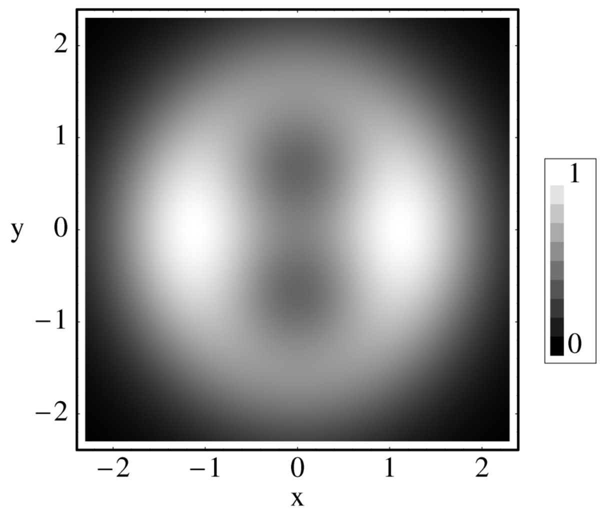

FIG. 2. The probability density for the superposition of the states $L = 2N$ and $L = 2N - 2$ for $N = 5$ with "dimples" reminiscent of vortex cores.

coordinate ζ_α. A complete set of *particle* states of angular momentum L is obtained using $n = L$ vortices:

$$\psi(\{z_i\}) = \int \prod_{\beta=1}^{n} d^2\zeta_\beta \, e^{-|\zeta|^2} \phi(\{\zeta_\alpha\}) \prod_{\beta=1}^{n} \prod_{i=1}^{N} (\zeta_\beta - z_i),$$

where $|\zeta|^2 = \sum_{\alpha=1}^{n} |\zeta_\alpha|^2$. This follows from the result [14] that an nth order polynomial in the z's may be expanded using the products of powers of the elementary symmetric polynomials, C_r, $0 \le r \le n$,

$$C_r(\{z_i\}) = \sum_{i_1 < i_2 < \cdots < i_r} z_{i_1} z_{i_2} \ldots z_{i_r}.$$

Although there is no need to write wave functions of *both* the particle, z_i, and the vortex coordinates, ζ_α, it will be convenient.

It will be useful to note the form of the resulting particle wave function, ψ, when there is one vortex. The construction implies the natural one-vortex states are $\phi_p(\zeta^*) = \zeta^{*p}/(p!\,\pi)$. The corresponding particle states are $\psi_{N-p} \propto C_p(\{z_i\})$ for $0 \le p \le N$, and, for $p > N$, $\psi_{N-p} = 0$. Note that the angular momentum of the particles is $L = N - p$, consistent with the special cases: $\psi_N = 1$, i.e., placing a vortex in this state has no effect on the nonrotating condensate; $\psi_{N-1} = \sum_{i=1}^{N} z_i = z_c$; and $\psi_0 = \prod_{i=1}^{N} z_i$, i.e., a "simple" single vortex.

It is tempting to relate the stable states at $L = n(N - m)$, to Eq. (3), as being n vortices in state $p = m$, with associated displacement of the vortex determined by $\langle|\zeta|^2\rangle_p = p + 1$. However, this implies $r_v^2 = p + 1$ and this leads to a contradiction unless $L \sim N^2$ (using the empirical relation $p = m \le n$), which is too restrictive.

Moreover, this purely vortex description, $\phi(\zeta)$, requires more vortices than n [in $L = n(N - m)$]. For example, $L = N$: expanding the product in Eq. (2) we see there is a term $z_c^N = C_1(\{z_i\})^N$ whose generation requires N vortex factors (even more for larger L). In addition, the number of vortices is not fixed, as the number in the vortex state ζ^N is indeterminate since they do not affect the particle wave function (in a sense it is the vortex vacuum state). This is in stark contrast to the incompressible ($\xi \longrightarrow 0$) case where the number of vortices is fixed and they are classical entities.

It might be supposed that, although there may be a fluctuating vortex population at large vortex quantum numbers, this is in the tail of the particle wave function and the description may be simple near the center of the trap. This is determined by computing the single-*vortex* density matrix, $\rho^{\rm v}(\zeta, \zeta')$, for the exact state $L = 2N - 2$, $N = 6$ (Fig. 3).

If the displaced vortex picture were correct, one might expect a factor in the vortex wave function of the form $(\zeta_1^* - \zeta_2^*)^2$ corresponding to the vortices rotating around the center of the trap. This factor alone would lead to the following eigenvalues, $\rho_m^{\rm v}$ (with corresponding eigenvectors ζ^m), of $\rho^{\rm v}$: $\rho_0^{\rm v} = \frac{1}{4}$, $\rho_1^{\rm v} = \frac{1}{2}$, and $\rho_2^{\rm v} = \frac{1}{4}$. As can be seen from Fig. 3, this is not the case. The most pronounced feature is a *maximum* at $m = 0$. The vortices tend to condense, in the $m = 0$ state, not to separate in $|\zeta_1^* - \zeta_2^*|$. (Further evidence comes from evaluating the particle density matrix [9].)

These difficulties in describing the $\psi_{n,m}$ states purely in vortex variables occur because the particles are *binding* to the vortices. This leads to a strongly correlated state whose description requires additional vortex variables if they are used alone. One interpretation uses ideas from the quantum Hall effect [12]: At the center of each vortex there is a decrease in the particle density. Thus, in terms of interparticle interactions, this is a low energy region for an additional particle.

Mathematically this is described most easily for the Laughlin state, using N vortices, ζ_α, with a factor $\prod_{i,i\neq\alpha}^{N}(\zeta_\alpha - z_i)$ where the αth particle experiences no suppression of its amplitude: it is "bound." This can be expressed as

$$\psi^{\rm Lau}(\{z_i\}) = \int \prod_{\beta=1}^{N} d^2\zeta_\beta \, e^{-|\zeta|^2} e^{\mathbf{z}\cdot\boldsymbol{\zeta}^*} \prod_{\alpha\neq j} (\zeta_\alpha - z_j).$$

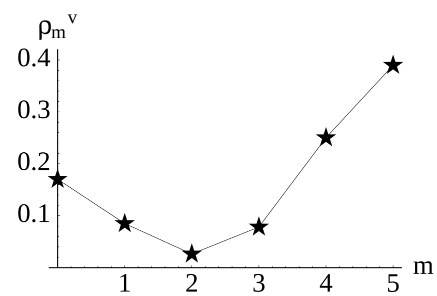

FIG. 3. Eigenvalues ($\rho_m^{\rm v}$) of the single *vortex* density matrix for eigenfunctions ζ^m for $L = 2N - 2$ for $N = 6$. The trace is normalized to unity, having suppressed the weight associated with the vortex "vacuum" state, $m = N$.

[Noting $e^{z\eta^*}$ and $(-1)^n \eta^{*n} e^{z\eta^*}$, respectively, play the roles [7,15] of a delta function and its nth derivative within the lowest Landau level.] That is, the delta function factors bind the ith particle to the ith vortex.

To generate the states $\psi_{n,m}$ we use the derivatives of the lowest Landau level delta function so that

$$\psi_{n,m}(\{z_i\}) = \int \prod_{\beta=1}^{N} d^2\zeta_\beta\, e^{-|\zeta|^2} e^{\mathbf{z}\cdot\boldsymbol{\zeta}^*} \times \phi_{n,m}(\{\zeta_\gamma^*\}) \prod_{\alpha\neq j} (\zeta_\alpha - z_j)\,,$$

where

$$\phi_{n,m}(\{\zeta_\alpha\}) = \prod_{\beta=1}^{N} \zeta_\beta^{*(N-1-n)} \sum_{\gamma_1<\gamma_2<\cdots<\gamma_m}^{N} \zeta_{\gamma_1}^{*n} \zeta_{\gamma_2}^{*n} \ldots \zeta_{\gamma_m}^{*n}\,,$$

which can be interpreted as a condensate of $(N - m)$ composites (each consisting of an atom and a vortex) in the state, ζ^{*r} with $r = (N - 1 - n)$. The remaining unbound atoms remain condensed in the single-particle ground state. [The states $L = n(N - m)$ are also selected using a composite fermion approach [16].]

The remaining stable states are consistent with the bosonic Pfaffian state [4,17], at $L = \frac{1}{2}N(N - 2)$ for even N and $L = \frac{1}{2}(N - 1)^2$ for odd N.

$$\psi^{\mathrm{Pf}}(\{z_i\}) = \prod_{i<j} (z_i - z_j)\, \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right),$$

where the Pfaffian is defined

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \mathcal{A}\left[\frac{1}{(z_1 - z_2)} \frac{1}{(z_3 - z_4)} \cdots \frac{1}{(z_{N-1} - z_N)}\right],$$

where $\mathcal{A}$ denotes antisymmetrization of the following product. (This is generalized for odd N by omitting one of the particles in each term of the antisymmetrization [14].) The overlaps of ψ^{Pf} with the exact ground state for $N = 5$ and $L = 8$, $N = 6$ and $L = 12$, and $N = 7$ and $L = 18$ are 0.91^2, 0.90^2, and 0.80^2.

Some nearby stable states, e.g., $L = 10$ and $L = 14$, are well described as simple modifications of the Pfaffian state. This uses the conjecture (which has been demonstrated by direct evaluation for $4 \le N \le 8$) that the Pfaffian state may be represented by a product of two Laughlin states for $N/2$ particles [or, for odd N, a cluster of $(N - 1)/2$ and one of $(N + 1)/2$]:

$$\psi^{\mathrm{Pf}}(\{z_i\}) = S \prod_{i<j\in\sigma_1} (z_i - z_j)^2 \prod_{k<l\in\sigma_2} (z_k - z_l)^2,$$

where the two subsets, σ_1 and σ_2, each have $N/2$ particles [$(N - 1)/2$ and $(N + 1)/2$ for odd N]. S indicates that the wave function is symmetrized over the distribution of the particles into these subsets. These two well-correlated clusters appear to be "dual" to the clusters of Halperin [18] which have a high internal energy, due to the lack of nodal factors.

For example, the state $N = 6$, $L = 14$ has overlap 0.96^2 with a state with two quanta of angular momenta in the center of mass motion of the clusters (defining $Z_b = \sum_{i\in\sigma_b} z_i$, $b = 1$ or 2):

$$\psi^{L=14}(\{z_i\}) = S(Z_1 - Z_2)^2 \psi^{\mathrm{Pf}}.$$

The state $N = 6$, $L = 10$ has overlap 0.97^2 with a state where there is one factor of center of mass motion and one vortex has been "removed" from one of the clusters:

$$\psi^{L=10}(\{z_i\}) = S(Z_1 - Z_2)\psi^{\mathrm{Pf}} \prod_{p\in\sigma_1} \sum_{q<p,q\in\sigma_1} \frac{1}{z_p - z_q}\,.$$

(The apparent asymmetry of the last factor involving only the first cluster, σ_1, is illusory due to the overall symmetrization.)

In conclusion, this Letter provides evidence that the weak coupling limit of rotating Bose-Einstein condensates contains some novel phenomena, even at moderate angular momenta.

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