Strong Anomaly in Diffusion Generated by Iterated Maps

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We investigate the diffusion generated deterministically by periodic iterated maps that are defined by $x_{t+1} = x_t + ax_t^z \exp[-(b/x_t)^{z-1}]$, $z > 1$. It is shown that the obtained mean squared displacement grows asymptotically as $\sigma^2(t) \sim \ln^{1/(z-1)}(t)$ and that the corresponding propagator decays exponentially with the scaling variable $|x|/\sqrt{\sigma^2(t)}$. This strong diffusional anomaly stems from the anomalously broad distribution of waiting times in the corresponding random walk process and leads to a behavior obtained for diffusion in the presence of random local fields. A scaling approach is introduced which connects the explicit form of the maps to the mean squared displacement.

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The concept of deterministic diffusion is already well established and has been observed in a broad range of chaotic dynamical systems, both conservative and dissipative [1–5]. Most of the examples display either normal diffusion, for which the mean squared displacement (MSD) grows linearly in time, or anomalous diffusion where the MSD follows asymptotically as

$$\sigma^2(t) \sim t^\nu, \tag{1}$$

with $\nu < 1$ for subdiffusion [1,4], and $\nu > 1$ for enhanced diffusion [3,4]. Such diffusional anomalies have been observed, among other systems, in amorphous semiconductors [6] and in polymer networks [7] for $\nu < 1$, and in rotating laminar fluid flows [8] for $\nu > 1$. Other examples include both numerical and experimental studies [9,10]. Another diffusional anomaly which has been less explored is the case where the MSD grows logarithmically in time, $\sigma^2(t) \sim \ln^\beta(t)$. This type of anomaly, which we refer to as strong anomaly, has been derived in classical models such as the Sinai model [11], in some aperiodic environments [12,13], and for random walks on bundled structures [14]. Interestingly, recent numerical simulations have shown that strong anomaly occurs often when a system, which otherwise displays simple subdiffusive behavior ($\nu < 1$), experiences an additional field [15].

One-dimensional maps are probably the simplest dynamical systems which produce normal and anomalous diffusion processes [1,3,4]. These maps generate sets of trajectories according to a rule

$$x_{t+1} = x_t + F(x_t) \tag{2}$$

with the following symmetry properties of $F(x)$: (i) $F(x)$ is periodic, with a periodicity interval set to 1, $F(x) = F(x + N)$, where N is an integer, and (ii) $F(x)$ has an inversion antisymmetry; namely $F(x) = -F(-x)$. Variations of these maps have been investigated by taking into account, for instance, quenched disorder [16], or an additional uniform bias [17] which break the symmetry of the map. Geisel and Thomae [1] considered a rather wide family of maps which obey

$$F(x) = ax^z, \quad \text{for } x \to +0, \tag{3}$$

with $z > 1$. Under certain conditions these maps lead to a MSD $\sigma^2(t) \equiv \langle x^2(t)\rangle \sim t^\nu$, where ν depends on the parameter z of the map, so that $z > 2$ leads to $\nu < 1$. The mechanism that generates this anomalously slow diffusion can be understood in terms of a random walk picture with a broad distribution of waiting times [1,4]. The symmetry properties described above, together with Eq. (3), define a translationally invariant discontinuous function. The latter corresponds to a series of cells given by $N - 0.5 \le x < N + 0.5$, where each cell has a fixed point at its center $x = N$. If we first restrict ourselves to the central cell with the fixed point at $x = 0$, then an injection at $x_i > 0$ starts an iteration inside the central cell. This iteration process comes to an end when the "transfer" region out of the cell is reached; namely when $x_t > 0.5$. Here a jump into a neighboring cell is performed and an iteration process in this new cell restarts. Depending on whether this injection has been on the right or the left side of the new fixed point, the iteration leads to an increasing or decreasing series of $\{x_t\}$. This results in a jump to the right or the left cell, respectively. The closer is the injection point to the fixed point, the longer is the residence time within the cell. Although the process itself is deterministic, the injection into the neighboring cells can be viewed as random. Since the injection point can be arbitrarily close to the fixed point of a cell, the residence time within a cell can be arbitrarily long. The process is described therefore as a random walk among cells with waiting times which are broadly distributed so that one can readily apply the continuous time random walk framework [4,6]. The parameter z determines the universality class of the maps given by Eqs. (2) and (3). The latter have been shown to lead to a power-law $t^{-z/(z-1)}$ behavior of the waiting time distribution (WTD). For $z > 2$ these WTDs have no finite moments and the corresponding MSD grows anomalously in time according to $\sigma^2(t) \sim t^{1/(z-1)}$ [1,4].

Our aim here it is to introduce a map which leads to a logarithmic growth with time of the MSD. In order to

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achieve this strong anomaly, $\sigma^2(t) \sim \ln^\beta(t)$, one needs iteration rules which result in WTDs that decay even more slowly than the power law mentioned above. The underlying idea might be to modify the maps in Eqs. (2) and (3) in a way that instead of dealing with a single exponent z, as in Eq. (3), which determines the MSD on all times scales according to Eq. (1), one should create a *hierarchy* of exponents, which depend on the distance x from the fixed point. Here we take advantage of the fact that in the anomalous case different spatial regimes of the map dominate different time scales of the MSD. This suggests the map in Eq. (2) with

$$F(x) = ax^z \exp[-(b/x)^{z-1}], \qquad (4)$$

for $0 \leq x < 0.5$, with $z > 1$, and $b > 0$. The map in Eq. (4) generates WTDs which are dominated by a logarithmic decay rather than by a simple power law. It reduces to the Geisel-Thomae map when $b = 0$. Figure 1 shows the map in Eqs. (2) and (4), for three successive cells. The extremely broad WTD is reflected in the fact that in the vicinity of the fixed points the slope is hardly distinguishable from the value 1. If the slope were indeed 1, then each iteration step would reproduce itself, which would correspond to an infinite trapping time within a cell.

In order to obtain from Eqs. (2) and (4) the waiting time $\tau(x_i)$ as a function of the injection point x_i, we turn the differences quotient $x_{t+1} - x_t = ax_t^z \exp[-(b/x_t)^{z-1}]$, which close to the fixed point is small, into the corresponding differential equation $dx/dt = ax^z \exp[-(b/x)^{z-1}]$. Since in the iteration process a cell is left when $x_t \geq \frac{1}{2}$, the waiting time $\tau(x_i)$ in the cell, as a function of the injection point x_i, is

$$\tau(x_i) = \int_{x_i}^{0.5} \frac{dx}{F(x)} = -T \int_{(b/x_i)^{z-1}}^{(2b)^{z-1}} e^y dy$$
$$\simeq T \exp[(b/x_i)^{z-1}], \qquad (5)$$

where $T = [ab^{(z-1)}(z-1)]^{-1}$. If one assumes no correlations between subsequent jumps from cell to cell, the distribution $\psi(t)$ of residence times can be derived from the distribution $\eta(x_i)$ of injection points x_i by $\psi(t)dt = \eta(x_i)dx_i$ [1]. In the limit of $t \to \infty$, $\eta(x_i) \to \eta(x_i = 0)$, independent of x_i. Calculating the differential $|dx_i/d\tau|$ by means of Eq. (5) yields asymptotically,

$$\psi(t) \sim \frac{T}{t \ln^{z/(z-1)}(t/T)}, \qquad z > 1. \qquad (6)$$

Similiar to the maps investigated earlier [1,4] this distribution does not possess finite moments. The condition $z > 1$ yields $z/(z-1) > 1$ and ensures the normalization of $\psi(t)$. Figure 2 displays the distribution of waiting times for $z = 1.25$, $z = 1.5$, and $z = 2$, as obtained by numerical iterations of the maps in Eqs. (2) and (4). The agreement with Eq. (6) is very good.

According to the continuous time random walk theory [6] the MSD $\sigma^2(t)$ can now be derived by means of the relation, in Laplace space [4,6,18],

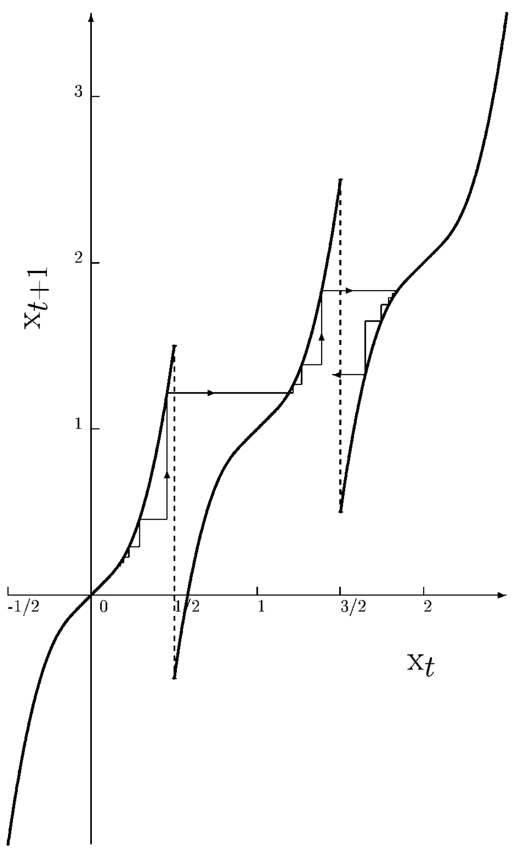

FIG. 1. The map $x_{t+1} = x_t + F(x_t)$, defined by Eq. (4) with $z = 2$, $b = 0.5$, and $a = 4e$.

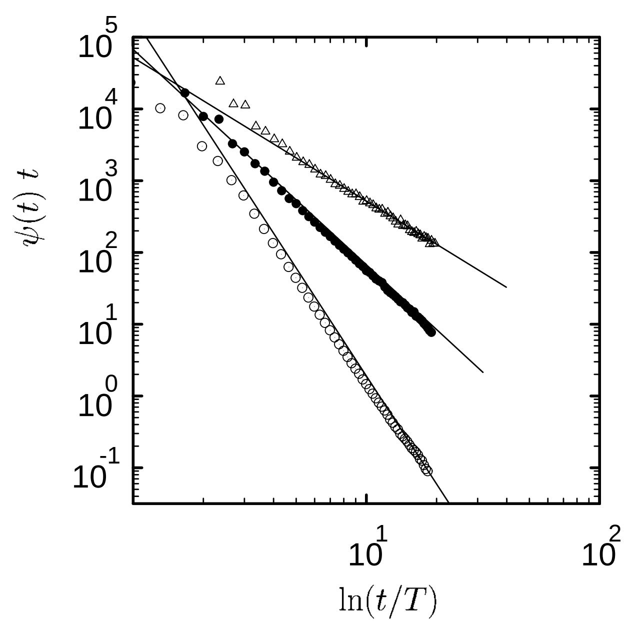

FIG. 2. Double-logarithmic plot of $\psi(t)\,t$ versus $\ln t$ for $z = 1.25$ (open circles), $z = 1.5$ (filled circles), and $z = 2$ (open triangles). The slopes of the lines correspond to the theoretical predictions $z/(z-1)$ of Eq. (6). In order to record the extremely broad distribution, we counted the waiting times in logarithmically equidistant time boxes. In the numerical simulations averages have been taken over 20 000 realizations. For each realization $t = 10^8$ iteration steps have been performed.

$$L\{\sigma^2(t)\} \sim \frac{\hat{\psi}(s)}{s[1 - \hat{\psi}(s)]}, \qquad (7)$$

where $L\{\sigma^2(t)\}$ denotes the Laplace transform of $\sigma^2(t)$ and $\hat{\psi}(s) = L\{\psi(t)\} = \int_0^\infty e^{-st}\psi(t)\,dt$. Starting from Eq. (6), one finds, following Havlin and Weiss [18], that

$$\hat{\psi}(s) \sim 1 - \frac{A}{[\ln(1/s)]^{z/(z-1)}}, \qquad s \to 0. \qquad (8)$$

The asymptotic time dependence of the MSD for $t \to \infty$ can be now derived from Eqs. (7) and (8) in the limit $s \to 0$ by using a Tauberian theorem [19], which yields

$$\sigma^2(t) \sim \ln^{1/(z-1)}(t/T). \qquad (9)$$

The map introduced by Eq. (4) drastically changes the power-law anomaly obtained by Geisel and Thomae, $t^{1/(z-1)}$, to a logarithmic strong anomaly, Eq. (9). Figure 3 shows the MSDs corresponding to $z = 1.25$, $z = 1.5$, and $z = 2$. Again, there is a good agreement with the continuous time random walk description. Interestingly, the case $z = 1.25$ in Figs. 2 and 3 leads to the behavior obtained in the Sinai model, namely, $\sigma^2(t) \sim \ln^4(t)$ [11].

Equation (9) can be also obtained on more general grounds. We argue that the time t needed to pass n cells is

$$t = \sum_{i=1}^{n} \tau_i = \sum_{i=1}^{n} \tau(x_i), \qquad (10)$$

where τ_i is the time spent in the ith cell. In the case that the WTD has a mean, $\lim_{n\to\infty}(\frac{1}{n}\sum_{i=1}^n \tau_i) \equiv \langle\tau\rangle < \infty$, one obtains a linear increase in n of the time t, $t = \langle\tau\rangle n$. If we assume the absence of long-range correlations, the number of jumps among the cells scales with the distance according to $n \sim x^2$, which results in normal diffusion, $\sigma^2(t) \sim t$.

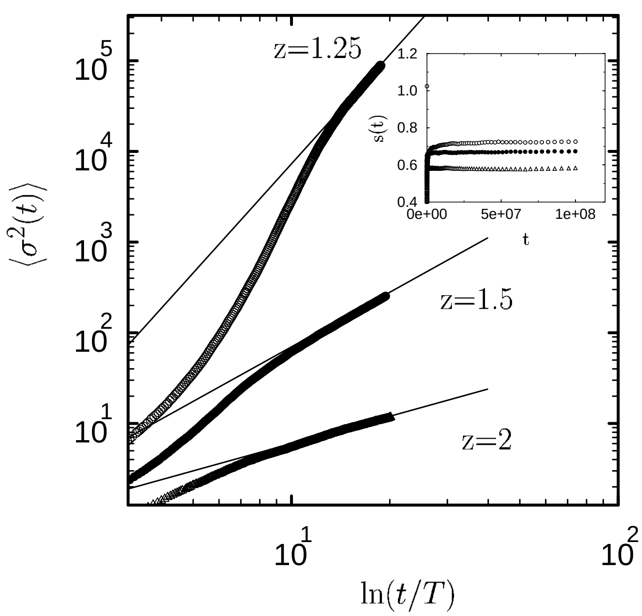

FIG. 3. The MSD $\sigma^2(t)$ plotted double logarithmically versus $\ln(t)$ for the three values of z in Fig. 2. The slopes of the lines correspond to the theoretical prediction of Eq. (9). Inset: $s(t) \equiv \sigma^2(t)/\ln^{1/(z-1)} t$ versus time t for the same values of z as in Fig. 2 represented by the same symbols.

When the waiting times τ_i originate from an anomalously broad distribution without a finite mean, the time t in Eq. (10) is dominated by the longest of the n waiting times, which is in turn determined by the injection point $x_{\min}$, which is the closest to the fixed point. Since the smallest of n random numbers, here given by the injection points x_i with $0 \le x_i \le 0.5$, is $x_{\min} \simeq 0.5/n$ [20], this yields $t \to T\exp[(b/x_{\min})^{z-1}] \simeq T\exp[(bn/\text{const})^{z-1}]$. Because of the relationship $n \sim \sigma^2(t)$, this leads to the time dependence of the MSD according to Eq. (9). Since $\tau(x_{\min})$ is related to $F(x)$ through the left side of Eq. (5), the above arguments can be summarized by

$$t \to T\int_{1/\sigma^2(t)}^{0.5} \frac{dx}{F(x)}. \qquad (11)$$

For $F(x) \sim x^z$ with $z > 2$, this scaling relation results in $\sigma^2(t) \sim t^\nu$ with $\nu = 1/(z-1)$, which is consistent with [1], while for $F(x)$ in Eq. (4) this results in Eq. (9).

In order to get a detailed insight into the dynamics generated by an iterated map of the type defined by Eq. (4), we derive the corresponding propagator $P(x,t)$, the probability that after t iterations the achieved distance is x. According to Eq. (5), for all iteration series with an injection point smaller than $x_{\text{in}}(t) \simeq b\ln^{1/(z-1)}(t/T)$ the trajectory resides within the cell during time t. As has been shown above, in this strongly non-self-averaging system, one can neglect all shorter waiting times in comparison with the longest one. Therefore, the particle is almost always trapped in the cell with the smallest injection point reached up to the considered time t. Then, the probability to be trapped after n jumps among the cells is

$$P(n,t) = [1 - W(t)]^{n-1}W(t) \simeq W(t)\exp[-B(t)n], \qquad (12)$$

where $B(t) = \ln\{1/[1 - W(t)]\}$ and where the sticking probability $W(t)$ of a single jump is equal to the probability that the corresponding injection takes place within the regime $N - x_{\text{in}}(t) < x < N + x_{\text{in}}(t)$ around the fixed point at $x = N$, which leads to $W(t) = 2x_{\text{in}}(t)$. This trapping approach takes advantage of the weak logarithmic dependence of $W(t)$ on t given by $x_{\text{in}}(t)$, which leads to a sticking probability that is unaffected by the time already spent in the preceding cells. In contrast to this, in the Geisel-Thomae map [1,4], $x_{\text{in}}(t)$ increases with a power of t, which introduces a memory which is reflected in the nonexponential decay. In order to express the propagator in terms of the distance x one should calculate the sum

$$P(x,t) = \sum_{n=0}^{\infty} \Phi(n\,|\,x)P(n,t), \qquad (13)$$

where $\Phi(n|x)$ is the distribution of performed cell jumps n at a given distance x which for $n \gg x^2$ is

$$\Phi(n|x) \simeq \Phi(0|x)\exp\left[\frac{-x^2}{2n}\right]. \qquad (14)$$

If we insert $W(t) = 2x_{\text{in}}(t)$ into Eq. (12), and after the transition to the continuum case apply the steepest decent method, we obtain

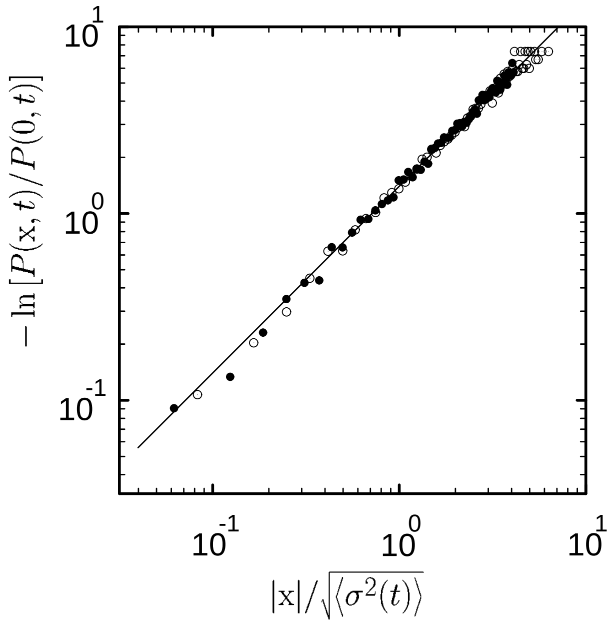

FIG. 4. The propagator in a scaling fashion according to Eq. (15) for two times, $t = 10^5$ (open circles) and $t = 10^8$ (filled circles). The theoretical slope 1, is given by the solid line.

$$P(x,t) = P(0,t)\exp[-\xi], \qquad \xi = |x| \Big/ \sqrt{\sigma^2(t)}, \quad (15)$$

with $P(0,t) = [2\sqrt{\sigma^2(t)}\,]^{-1}$. The exponential behavior of $P(x,t)$ in Eq. (15) strongly differs from the propagator that corresponds to the Geisel-Thomae map, where $P(x,t) \sim \exp(-[|x|/\sqrt{\sigma^2(t)}\,]^{2/(2-\nu)})$ with $\nu = 1/(z-1)$ [4]. The exponential behavior is reminiscent of the short time scaling behavior of the one-dimensional propagator calculated within the continuous time random walk for power law WTDs [21]. The propagator in Eq. (15) can be also derived from the WTD, Eq. (6), within the continuous time random walk formulation [18].

Figure 4 shows the decay of the propagator for two different times in a scaling fashion for $z = 1.5$. The data collapse strongly supports the scaling prediction of Eq. (15) and the slope is in excellent agreement with the theoretical value 1. The simple exponential decay of the propagator has also been analytically derived for the Sinai model [22], where $\sigma^2(t) \sim \ln^4(t)$, and has been found in biased diffusion on percolation [23]. In the latter case recent numerical results yield $\sigma^2(t) \sim \ln^{\gamma(\epsilon)}(t)$ where the parameter ϵ determines the strength of the field [23]. Our maps generate diffusion of the Sinai-type (MSD and propagator) with a broad range of exponents $1/(z-1)$, $z > 1$. If one applies scaling arguments, as used in [24] for the Sinai-model, one can show that this new class of dispersive maps can be regarded also as simple models which show a $1/f$ power spectrum with $(\ln f)^{1/(z-1)}$ corrections.

In summary, we have introduced a new family of periodic maps which lead to strong anomaly in the generated diffusional process. The MSD $\sigma^2(t)$ has been shown to grow logarithmically in time, $\sigma^2(t) \sim \ln^{1/(z-1)}(t)$, where the parameter z determines the universality class of the map, and the propagator is exponential in the scaling variable $|x|/\sqrt{\sigma^2(t)}$. This behavior is of the Sinai-type anomaly. In particular, we obtain the Sinai result [11] for $z = 5/4$.

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