# Adaptation to the Edge of Chaos in the Self-Adjusting Logistic Map

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Self-adjusting, or adaptive, systems have gathered much recent interest. We present a model for self-adjusting systems which treats the control parameters of the system as slowly varying, rather than constant. The dynamics of these parameters is governed by a low-pass filtered feedback from the dynamical variables of the system. We apply this model to the logistic map and examine the behavior of the control parameter. We find that the parameter leaves the chaotic regime. We observe a high probability of finding the parameter at the boundary between periodicity and chaos. We therefore find that this system exhibits adaptation to the edge of chaos.

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Self-adjusting, or adapting, systems are ubiquitous in nature. Living systems are constantly changing their own properties in response to their environment. For example, the Asian firefly will adjust its flashing frequency to match the frequency of outside stimuli, such as the flashing of another firefly [1]. Because of the importance of adaptation in natural systems, there have been many studies which try to characterize evolutionary, adaptive behavior [2,3]. Many of these studies show that adaptive systems will adapt to a new state at the boundary of chaos and order, called the edge of chaos. Packard [4] showed that this effect occurred for a population of cellular automata rules evolving with a genetic algorithm. Pierre and Hübler [5] studied two competitive, adaptive agents which used both control and modeling to predict the behavior of the logistic map and found that, over time, the agents use a control which places the logistic map at the edge of chaos. The edge of chaos also occupies a prominent position because it has been found to be not only an optimal setting for control of a system [6], but also an optimal setting under which a physical system can support primitive functions for computation [7].

We suggest a new model for self-adjusting systems, as proposed by Ritz and Hübler [6]. An adjustable system is a system where the control parameters of the system are not constant in time. Control parameters are distinguished from dynamical variables through a separation of time scales, i.e., the control parameters vary much more slowly than do the dynamical variables [8]. The dynamics of the control parameters is simple, overdamped motion without an attractor. If the forcing function for the parameter depends only on the system itself, the system is called self-adjusting. We introduce a self-adjusting logistic map, using a forcing function which is a low-pass filtered feedback from the dynamical variables. We find that this map exhibits adaptation to the edge of chaos.

The logistic map has a dynamical variable $x_n$ and parameter $a$ and is a function of time, $n$:

$$x_{n+1} = ax_n(1 - x_n), \qquad 0 \leq x_n \leq 1, \qquad 0 \leq a \leq 4. \tag{1}$$

The parameter $a$ determines the type of dynamics which occurs for the dynamical variable, $x_n$ [9]. If $0 < a < 3$, then the dynamics of $x_n$ has a stable fixed-point attractor. For intermediate values, $3 < a < 3.569$ the dynamics of $x_n$ is periodic. For $3.569 < a < 4$, the dynamics of $x_n$ is mostly chaotic. There are, however, values of the parameter in this range which lead to periodic behavior of $x_n$. These values are called periodic windows. $a = 3.84$ is contained in the well-known period 3 window. The edge of chaos refers to values of $a$ which lead to periodic (chaotic) behavior in $x_n$ and with only a small change would lead to chaotic (periodic) $x_n$ dynamics. Thus, values of $a$ which are very near to 3.569, or are very near to the periodic windows, are at the edge of chaos.

If the parameter, $a$, changes slowly with time, as in

$$a_{n+1} = a_n + \epsilon f_n, \qquad 0 \leq a_n \leq 4,$$
$$n = 0, 1, 2, \ldots, \tag{2}$$

where $f_n$ is a forcing function, and $\epsilon$ is a small constant, the logistic map becomes an adjustable system. If the forcing, $f_n$, is a function, $g$, of only the dynamical variable, $x_n$, the logistic map is self-adjusting. Because of the requirement of a separation of time scales, a low-pass filter is a logical choice. By damping out the high frequency terms, both the requirements of overdamped motion and separation of time scales can be achieved. In addition, low-pass filters are common in natural and experimental situations.

The low-pass filtering can be achieved in numerical simulations by a Fourier analysis of the time series for $x_n$. If $N$ time steps are used, the Fourier sine and cosine coefficients are given by $\beta_{n0} = (1/N)\sum_{t=0}^{N-1} x(t + n - N + 1)$, $\alpha_{nk} = (2/N)\sum_{t=0}^{N-1} x(t + n - N + 1) \times \sin(2\pi kt/N)$, and $\beta_{nk} = (2/N)\sum_{t=0}^{N-1} x(t + n - N + 1)\cos(2\pi kt/N)$ for $k = 1, 2, \ldots, (N - 1)/2$ where $k$ is the frequency. If $N$ is odd, an extra term is

needed: $\beta_{n(N+1)/2} = (1/N) \sum_{t=0}^{N-1} x(t + n - N + 1) \times \cos[\pi(N + 1)t/N]$.

A low-pass filter with dc cutoff and a very low frequency cutoff would keep only terms $\alpha_{n1}$ and $\beta_{n1}$. The back transformation would then become

$$\bar{x}n = \alpha_{n1} \sin\left(\frac{2\pi n}{N}\right) + \beta_{n1} \cos\left(\frac{2\pi n}{N}\right). \qquad (3)$$

If the forcing is only applied once every $N$ steps, and is evaluated when $n$ is a multiple of $N$, $f_n$ becomes simply

$$f_n = \begin{cases} \epsilon \bar{x}_N = \epsilon \beta_{n1}, & \text{if } n = iN, \\ 0, & \text{if } n \neq iN, \end{cases} \quad i = 1, 2, 3, \ldots. \qquad (4)$$

Numerical simulations of the self-adjusting logistic map were performed. $N \gg 1$ and $\epsilon \ll 1$ were used to ensure a good separation of time scales. Figure 1 shows the time dependence of three different initial parameter values. For $a_0 = 3.5$, there is no change in $a$ with time. The limiting dynamics of $x_n$ when $a = 3.5$ is periodic. However, the dynamics of $a$ for both the initial values $a_0 = 3.8$ and $a_0 = 3.9$ shows a ragged time dependence until a value of $a$ is reached that leads to a periodic limiting dynamics for $x_n$. The system leaves the chaotic regime and settles on a periodic dynamics. The limiting value of $a$ leads to periodic behavior in $x_n$. However, only a small change in this limiting value is necessary to create chaotic dynamics in $x_n$. Therefore we say that the system has adapted to the edge of chaos.

To illustrate adaptation, 300 initial values of the parameter, $a_0$, were taken evenly over the interval [3.4:4]. Figure 2 shows a histogram for the distribution of parameter values for two times, $n = 0$ and $n = 60\,000$. As can be seen, the initial distribution is flat over the interval [3.4,4].

At $n = 60\,000$, the probability is very small for values of the parameter, $a$, whose limiting dynamics is chaotic. The probability is very high, however, for those values of $a$ which have a limiting dynamics which is periodic. In most cases, the system has evolved to the periodic windows of the system, which are labeled in the figure. This high probability at values of $a$ which are at the edge of chaos is an alternative description of adaptation to the edge of chaos, and is satisfied by this system. Initial parameter values which lead to periodic behavior have not changed. This observation leads to an approximation for the behavior of the low-pass filtered dynamics, $\bar{x}_n$:

$$\bar{x}_n(a) \approx \begin{cases} \delta_n, & \text{if } a \text{ leads to chaotic } x \text{ dynamics,} \\ 0, & \text{if } a \text{ leads to periodic } x \text{ dynamics,} \end{cases} \qquad (5)$$

where $\delta_n$ is a nonzero number. Equation (5) can be understood in terms of the recurrence time and power spectrum of the $x_n$ dynamics of the logistic map. Periodic behavior has, by definition, a finite recurrence time. This leads to a power spectrum which has a lowest frequency $\omega_0$, which is proportional to the inverse of the recurrence time. Chaotic dynamics, however, has an infinite recurrence time and thus its power spectrum does not have a lowest frequency component [10]. Therefore, if the cutoff frequency of the low-pass filter is $\omega_c$, a condition on the low-passed dynamics can be made:

$$\bar{x}_n(a) = 0 \quad \text{if } \omega_c < \omega_0(a), \qquad (6)$$

where $\omega_0(a)$ is the lowest frequency of the $x_n$ dynamics with parameter value $a$.

To understand the ragged time dependence in Fig. 1, we look at the autocorrelation function, $C$, of the feedback, $f_n$:

$$C(j) \equiv \langle f_n f_{n+j} \rangle = \frac{1}{S} \sum_{n=0}^{n=S} f_n f_{n+j}, \qquad (7)$$
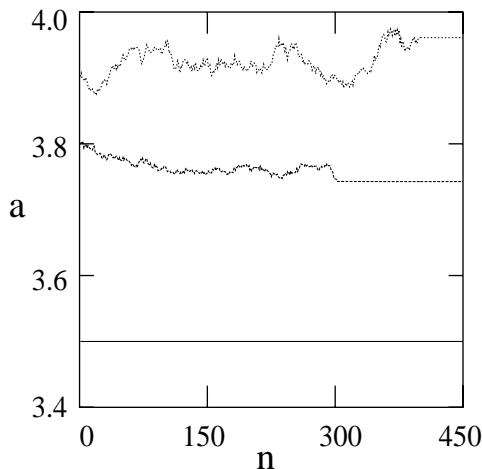


FIG. 1. Time dependence of three initial parameter values. $a = 3.5$ corresponds to periodic motion, while $a = 3.8$ and $a = 3.9$ correspond to chaotic motion. The final value of $a_0 = 3.8$ is 3.74 (period 5) and the final value of $a_0 = 3.9$ is 3.96 (period 4). For this simulation, $N = 20$ and $\epsilon = 0.1$ were used.
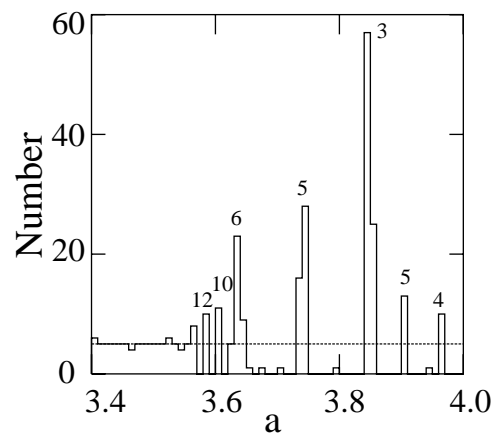


FIG. 2. Distributions of parameter values between 3.4 and 4. The initial distribution at $n = 0$ is flat, and the final distribution at $n = 60\,000$ clearly shows a high probability at the edge of chaos, in the periodic windows. The period of each window is labeled above the corresponding peak. $N = 20$ and $\epsilon = 0.1$ were used in this simulation.
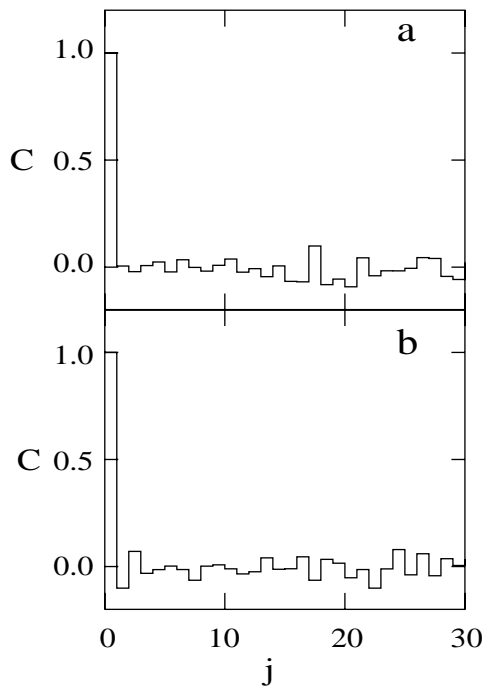
FIG. 3. (a) shows the autocorrelation function for a random data set, while (b) shows the autocorrelation function for the feedback, $f_n$. Both functions are normalized to $C(0)$. $N = 20$ and $\epsilon = 0.1$ were used.

where $S$ is the number of time steps taken while the parameter is still changing. A random data set will be $\delta$ correlated, as in

$$\frac{C(j)}{C(0)} = \begin{cases} 1, & j = 0, \\ 0, & j \neq 0. \end{cases} \qquad (8)$$

A comparison of the autocorrelation function of a random data set and the feedback, $f_n$, is shown in Fig. 3. As can be seen, the feedback is very nearly $\delta$ correlated, which means that the ragged time dependence observed in Fig. 1 is a diffusive, random walk motion.

We have shown that, for the self-adjusting logistic map, initially chaotic states adapt to periodic states which are at the edge of chaos. Our model uses a low-pass filtered feedback from the dynamical variables to the parameter of the system. This approach is different from previous studies which have used cellular automata, genetic algorithms, or neural networks to drive the adaptation. Our model is much simpler, using only feedback for adaptation. A simple, feedback-based model is more applicable to many physical systems which do not have gene codes or memories. We feel that adaptation to the edge of chaos is a generic property of systems with a low-pass filtered feedback, independent of both the form of the low-pass filter and the specific system under study. The low-pass filtered feedback exploits basic properties of periodicity and chaos, and so adaptation toward the edge of chaos should be a common property of such systems.

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