Opening up Extra Dimensions at Ultralarge Scales

Ruth Gregory,[1] Valery A. Rubakov,[2] and Sergei M. Sibiryakov[2]
[1]*Centre for Particle Theory, Durham University, South Road, Durham, DH1 3LE, United Kingdom*
[2]*Institute for Nuclear Research of the Russian Academy of Sciences, 60th October Anniversary Prospect, 7a, Moscow 117312, Russia*
(Received 4 April 2000)

The standard picture of viable higher-dimensional theories is that direct manifestations of extra dimensions occur at short distances only, whereas long-distance physics is effectively four-dimensional. We show that this is not necessarily true in models with infinite extra dimensions. As an example, we consider a five-dimensional scenario with three 3-branes in which gravity is five dimensional at both short- *and* very long-distance scales, with conventional four-dimensional gravity operating at intermediate length scales. A phenomenologically acceptable range of validity of four-dimensional gravity extending from microscopic to cosmological scales is obtained without strong fine-tuning of parameters.

PACS numbers: 11.10.Kk, 04.50.+h, 11.25.Mj

Our world is often thought to have more than four fundamental space-time dimensions. This point of view is strongly supported by string/M-theory, and higher-dimensional theories are currently being developed in various directions. The standard lore is that, in phenomenologically viable models, extra dimensions open up at short distances only, whereas above a certain length scale, and all the way up to infinite distances, physics is described by effective four-dimensional theories. In this Letter we show that the latter picture is not universal: there are models in which extra dimensions open up both at short *and very long* distances. Namely, gravity may become higher dimensional in both of these extremes. At intermediate length scales, physics is described by conventional four-dimensional laws, so the phenomenology of these models is still acceptable.

The starting point for our discussion is the observation [1] that extra dimensions may be infinite, with the usual matter residing on a 3-brane embedded in higher-dimensional space. In the original Randall-Sundrum (RS) model [1], gravity is effectively four dimensional at large scales due to the existence of a graviton bound state localized near the 3-brane in five dimensions. We point out that in other five-dimensional models with an infinite extra dimension (in particular, in the model introduced in Ref. [2]), localization of the graviton may be incomplete, and it is this property that leads to the restoration of the five-dimensional form of gravity at very long distances. The latter feature is manifest both in the case of static sources as well as in the dynamics of gravitational waves. It is likely that these phenomena are not peculiar to five-dimensional theories and occur in a number of models with more than one extra dimension.

It has been found recently [3] that exotic large-distance effects may also appear in models with compact extra dimensions, due to the possible presence of very light Kaluza-Klein states. This interesting scenario is, however, considerably different from ours in that the extra dimensions show up at large distances somewhat indirectly through the spectrum of the corresponding four-dimensional effective theory, whereas in our case, physics at ultralarge distances is intrinsically five dimensional.

As a concrete example, let us consider the five-dimensional model of Ref. [2]. The model contains one brane with tension $\sigma > 0$ and (for Z_2 symmetry) two branes with equal tensions $-\sigma/2$ placed at equal distances to the right and left of the positive tension brane in the fifth direction, z. We assume that conventional matter resides on the positive tension brane, and in what follows we will be interested in gravitational interactions of this matter. By Z_2 symmetry, we will henceforth consider only the region to the right of the positive tension brane.

The physical setup is as follows: The bulk cosmological constant between the branes, Λ, is negative as in the RS model; however, in contrast to that model, it is *zero* to the right of the negative tension brane. With appropriately tuned Λ, there exists a solution to the five-dimensional Einstein equations for which both positive and negative tension branes are at rest at $z = 0$ and $z = z_c$, respectively, z_c being an arbitrary constant. The metric of this solution is

$$ds^2 = a^2(z)\eta_{\mu\nu}dx^\mu dx^\nu - dz^2, \tag{1}$$

where

$$a(z) = \begin{cases} e^{-kz}, & 0 < z < z_c\,, \\ e^{-kz_c} \equiv a_-, & z > z_c\,. \end{cases} \tag{2}$$

The constant k is related to σ and Λ as follows: $\sigma = \frac{3k}{4\pi G_5}$, $\Lambda = -6k^2$, where G_5 is the five-dimensional Newton constant. The four-dimensional hypersurfaces $z = \text{const}$ are flat; the five-dimensional space-time is flat to the right of the negative-tension brane and anti–de Sitter between the branes. The space-time to the left of the positive tension brane is of course a mirror image of this setup.

This background has two length scales, k^{-1} and $\zeta_c \equiv k^{-1}e^{kz_c}$. We consider the case of large enough z_c, in which

 0031-9007/00/84(26)/5928(4)$15.00

the two scales are well separated, $\zeta_c \gg k^{-1}$. We see that gravity in this model is effectively four dimensional at distances r belonging to the interval $k^{-1} \ll r \ll \zeta_c(k\zeta_c)^2$, and is five dimensional *both* at short distances, $r \ll k^{-1}$ (this situation is exactly the same as in the RS model), and at long distances, $r \gg \zeta_c(k\zeta_c)^2$. In the latter regime of very long distances the five-dimensional gravitational constant gets effectively renormalized and no longer coincides with G_5.

To find the gravity law experienced by matter residing on the positive tension brane, let us study gravitational perturbations about the background metric (1). We will work in the Gaussian normal (GN) gauge, $g_{zz} = -1$, $g_{z\mu} = 0$. The linearized theory is described by the metric

$$ds^2 = a^2(z)\eta_{\mu\nu}dx^\mu dx^\nu + h_{\mu\nu}(x,z)dx^\mu dx^\nu - dz^2. \tag{3}$$

There are two types of linearized excitations in this model. One of them is a four-dimensional scalar—the radion—that corresponds to the relative motion of the branes (see, e.g., Refs. [2,4,5]). The wave function of the radion is localized between the branes, and, if the distance between the branes is not stabilized, the radion has zero four-dimensional mass and gives rise to the usual $1/r$ potential in an effective four-dimensional theory. In the particular model under discussion, the radion excitation has been studied in Ref. [2]. It is conceivable that the distance between the branes may be stabilized (cf. Ref. [6]); in which case the interactions due to the radion switch off at large distances.

In this Letter we are interested in excitations which leave the branes at rest, namely, the five-dimensional gravitons. In this case there exists a frame which is universally GN, in which the transverse-traceless gauge can be chosen, $h_\mu^\mu = 0$, $h_{\mu,\nu}^\nu = 0$, and the linearized Einstein equations take the same simple form for all components of $h_{\mu\nu}$,

$$\begin{cases} h'' - 4k^2 h - \frac{1}{a^2}\Box^{(4)}h = 0, & 0 < z < z_c\,, \\ h'' - \frac{1}{a_-^2}\Box^{(4)}h = 0, & z > z_c\,. \end{cases} \tag{4}$$

The Israel junction conditions on the branes are

$$\begin{cases} h' + 2kh = 0 & \text{at } z = 0 \\ [h'] - 2kh = 0 & \text{at } z = z_c \end{cases}, \tag{5}$$

where $[h']$ is the discontinuity of the z derivative of the metric perturbation at z_c, and four-dimensional indices are omitted. A general perturbation is a superposition of modes, $h = \psi(z)e^{ip_\mu x^\mu}$ with $p^2 = m^2$, where ψ obeys the following set of equations in the bulk:

$$\begin{cases} \psi'' - 4k^2\psi + \frac{m^2}{a^2}\psi = 0, & z < z < z_c\,, \\ \psi'' + \frac{m^2}{a_-^2}\psi = 0, & z > z_c\,, \end{cases} \tag{6}$$

with the junction conditions (5) (replacing h by ψ). It is straightforward to check that there are no negative modes, and indeed no normalizable solutions with $m^2 \geq 0$. The spectrum is therefore continuous, beginning at $m^2 = 0$. Explicitly, these modes are

$$\psi_m = \begin{cases} C_m[N_1(\frac{m}{k})J_2(m\zeta) - J_1(\frac{m}{k})N_2(m\zeta)], & 0 < z < z_c\,, \\ A_m\cos[\frac{m}{a_-}(z - z_c)] + B_m\sin[\frac{m}{a_-}(z - z_c)], & z > z_c\,, \end{cases} \tag{7}$$

where N and J are the Bessel functions, and $\zeta = \frac{1}{k}e^{kz}$. The constants A_m, B_m, and C_m obey two relations due to the junction conditions at the negative tension brane:

$$A_m = C_m\left[N_1\left(\frac{m}{k}\right)J_2(m\zeta_c) - J_1\left(\frac{m}{k}\right)N_2(m\zeta_c)\right], \tag{8a}$$

$$B_m = C_m\left[N_1\left(\frac{m}{k}\right)J_1(m\zeta_c) - J_1\left(\frac{m}{k}\right)N_1(m\zeta_c)\right]. \tag{8b}$$

The remaining overall constant C_m is obtained from the normalization condition. The latter is determined by the explicit form of Eq. (6) and reads

$$\int \psi_m^*(z)\psi_{m'}(z)\,\frac{dz}{a^2(z)} = \delta(m - m')\,. \tag{9}$$

The normalization factor is determined by the asymptotic behavior of ψ_m at $z \to \infty$, as is generally the case for continuum modes. One finds

$$\frac{\pi}{a_-}(|A_m|^2 + |B_m|^2) = 1 \tag{10}$$

which fixes C_m from (8a) and (8b).

It is instructive to consider two limiting cases. At $m\zeta_c \gg 1$ we obtain, by making use of the asymptotics of the Bessel functions,

$$C_m^2 = \frac{m}{2k}\left[J_1^2\left(\frac{m}{k}\right) + N_1^2\left(\frac{m}{k}\right)\right]^{-1} \tag{11}$$

which coincides, as one might expect, with the normalization factor for the massive modes in the RS model. In the opposite case $m\zeta_c \ll 1$ (notice that this automatically implies $m/k \ll 1$), the expansion of the Bessel functions in Eqs. (8a) and (8b) yields

$$C_m^2 = \frac{\pi}{(k\zeta_c)^3}\left(1 + \frac{4}{(m\zeta_c)^2(k\zeta_c)^4}\right)^{-1}. \tag{12}$$

It is now straightforward to calculate the static gravitational potential between two unit masses placed on the positive-tension brane at a distance r from each other, which is generated by the exchange of the massive modes (cf. Refs. [1,7])

$$V(r) = G_5\int_0^\infty dm\,\frac{e^{-mr}}{r}\,\psi_m^2(z = 0)\,. \tag{13}$$

It is convenient to divide this integral into two parts,

$$V(r) = G_5 \int_0^{\zeta_c^{-1}} dm \, \frac{e^{-mr}}{r} \, \psi_m^2(0) + G_5 \int_{\zeta_c^{-1}}^{\infty} dm \, \frac{e^{-mr}}{r} \, \psi_m^2(0) . \qquad (14)$$

At $r \gg k^{-1}$, the second term in Eq. (14) is small and it is similar to the contribution of the continuum modes to the gravitational potential in the RS model. It gives short distance corrections to Newton's law,

$$\Delta V_{\text{short}}(r) \sim \frac{G_5}{kr^3} = \frac{G_N}{r} \frac{1}{k^2 r^2} , \qquad (15)$$

where $G_N = G_5 k$ is the four-dimensional Newton constant.

Of greater interest is the first term in Eq. (14) which dominates at $r \gg k^{-1}$. Substituting the normalization factor (12) into this term (since $m < \zeta_c^{-1}$), we find

$$V(r) = \frac{G_5}{r} \int_0^{\zeta_c^{-1}} dm \, \frac{4e^{-mr}}{\pi m^2 k \zeta_c^3} \left(1 + \frac{4}{(m\zeta_c)^2 (k\zeta_c)^4}\right)^{-1} . \qquad (16)$$

This integral is always saturated at $m \lesssim r_c^{-1} \ll \zeta_c^{-1}$, where

$$r_c = \zeta_c (k\zeta_c)^2 \equiv k^{-1} e^{3kz_c} . \qquad (17)$$

Therefore, we can extend the integration to infinity and obtain

$$V(r) = \frac{G_N}{r} \frac{2}{\pi} \int_0^{\infty} dx \, \frac{e^{-(2r/r_c)x}}{x^2 + 1} = \frac{2G_N}{\pi r} [\text{ci}(2r/r_c) \sin(2r/r_c) - \text{si}(2r/r_c) \cos(2r/r_c)] \qquad (18)$$

where $x = mr_c/2$, and $\text{ci}/\text{si}(t) = -\int_t^{\infty} \frac{\cos/\sin(u)}{u} du$ are the sine and cosine integrals. We see that $V(r)$ behaves in a peculiar way. At $r \ll r_c$, the exponential factor in Eq. (18) can be set equal to 1 and the four-dimensional Newton law is restored, $V(r) = G_N/r$. Hence, at intermediate distances, $k^{-1} \ll r \ll r_c$, the collection of continuous modes with $m \sim r_c^{-1}$ has the same effect as the graviton bound state in the RS model. However, in the opposite case, $r \gg r_c$, we find

$$V(r) = \frac{G_N r_c}{\pi r^2} \qquad (19)$$

which has the form of "Newton's law" of five-dimensional gravity with a renormalized gravitational constant.

It is clear from Eq. (18) that at intermediate distances, $k^{-1} \ll r \ll r_c$, the four-dimensional Newtonian potential obtains not only short-distance corrections, Eq. (15), but also long-distance ones, $V(r) = G_N/r + \Delta V_{\text{short}}(r) + \Delta V_{\text{long}}(r)$. The long-distance corrections are suppressed by r/r_c, the leading term being

$$\Delta V_{\text{long}}(r) = \frac{G_N}{r} \frac{r}{r_c} \frac{4}{\pi} \left(\ln \frac{2r}{r_c} + \mathbf{C} - 1\right), \qquad (20)$$

where $\mathbf{C}$ is the Euler constant. The two types of corrections, Eqs. (15) and (20), are comparable at roughly $r \sim \zeta_c$. At larger r, deviations from the four-dimensional Newton's law are predominantly due to the long-distance effects.

In our scenario, the approximate four-dimensional gravity law is valid over a finite range of distances. Without strong fine-tuning however, this range is large, as required by phenomenology. Indeed, the exponential factor in Eq. (17) leads to a very large r_c even for microscopic separations, z_c, between the branes. As an example, for $k \sim M_{\text{Pl}}$ we only require $z_c \sim 50 l_{\text{Pl}}$ to have $r_c \sim 10^{28}$ cm, the present horizon size of the Universe, i.e., with mild assumptions about z_c, the four-dimensional description of gravity is valid from the Planck to cosmological scales (in this example, short-distance corrections are significant for $r \lesssim \zeta_c \sim 10^{-13}$ cm).

An interesting consequence of the incomplete localization of graviton is the dissipation into the fifth dimension of gravitational waves propagating to large distances. Let us consider gravitational waves generated by a periodic point-like source on the brane, $T(x, z) = T(\mathbf{x}) e^{-i\omega t} \delta(z)$, where the four-dimensional indices are again omitted. The gravitational field on the brane is given by the convolution of the source, $T(\mathbf{x})$, with the Green's function

$$G(\mathbf{x} - \mathbf{x}'; \omega) = 8\pi G_5 \int d(t' - t) \times G(x, x'; z = z' = 0) e^{-i\omega(t-t')} . \qquad (21)$$

Here the five-dimensional gravitational constant is included for convenience of comparison with four-dimensional formulas, and $G(x, x'; z, z')$ is the retarded Green's function of the linearized Einstein equations. The latter is constructed from the full set of eigenmodes in the usual way:

$$G(x, x'; z, z') = \int_0^{\infty} dm \, \psi_m(z) \psi_m(z') \times \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(x-x')}}{m^2 - p^2 - i\epsilon p^0} . \qquad (22)$$

After substitution of (22) into (21) and simplifications we get

$$G(\mathbf{x} - \mathbf{x}'; \omega) = \frac{2G_5}{r} \int_0^{\infty} dm \, \psi_m^2(0) e^{ip_\omega r} , \qquad (23)$$

where $r = |\mathbf{x} - \mathbf{x}'|$, $p_\omega = \sqrt{\omega^2 - m^2}$ when $m < \omega$, and $p_\omega = i\sqrt{m^2 - \omega^2}$ when $m > \omega$. We see that the gravitational field on the brane has the form of a superposition of massive four-dimensional modes. Only modes with $m < \omega$ are actually radiated; the other ones exponentially fall off from the source. Thus, as long as we are interested in gravitational waves, we can integrate in (23) only up to $m = \omega$.

Let us study the case $r_c^{-1} \ll \omega \ll k$. Then, the main contribution to (23) is given by modes with $m \sim r_c^{-1}$, whereas modes with larger masses give rise to corrections suppressed by ω/k [cf. (14) and (15)]. In this region of m, the eigenfunctions ψ_m are given by the explicit expressions (7) and (12), and p_ω is approximated by $\omega - \frac{m^2}{2\omega}$. In this way we get [cf. (18)]

$$G(\mathbf{x} - \mathbf{x}'; \omega) = \frac{2G_N}{r} e^{i\omega r} \frac{2}{\pi} \int_0^\infty dx \frac{e^{-i(2r/\omega r_c^2)x^2}}{1 + x^2}, \tag{24}$$

where again $x = mr_c/2$ and we extended the integration to infinity. This integral is expressed in terms of the error function:

$$G(\mathbf{x} - \mathbf{x}'; \omega) = \frac{2G_N}{r} e^{i\omega r}[1 - \text{erf}(\beta)]e^{\beta^2}, \tag{25}$$

where $\beta = e^{i(\pi/4)}\sqrt{\frac{2r}{\omega r_c^2}}$. When $\beta \ll 1$ (that is, $r \ll \omega r_c^2$), one has $\text{erf}(\beta) \approx 0$, and we obtain the usual $1/r$ dependence of the gravity wave amplitude on the distance to the source. In the opposite case $\beta \gg 1$ ($r \gg \omega r_c^2$), we make use of $\text{erf}(\beta) = 1 - \frac{1}{\sqrt{\pi}\beta} e^{-\beta^2}$ to obtain

$$G(\mathbf{x} - \mathbf{x}'; \omega) = \frac{2G_N}{r^{3/2}} \sqrt{\frac{\omega r_c^2}{2\pi}} e^{i\omega r - i(\pi/4)}, \tag{26}$$

that is, the amplitude is proportional to $r^{-3/2}$. Thus, the gravitational waves dissipate into the fifth dimension and, from the point of view of a four-dimensional observer on the brane, energy of gravity waves is not conserved.

This effect becomes considerable after the wave travels the distance of order $r \sim \omega r_c^2$ from the source. Note that at $\omega \gg r_c^{-1}$ this distance is much larger than the distance r_c at which the violation of four-dimensional Newton's law is appreciable. This difference between the two distance scales can in fact be seen to be a relativistic effect. The collection of five-dimensional graviton states with $m \sim r_c^{-1}$ may be viewed as an RS bound state which becomes metastable in our model. Since the only relevant parameter here is r_c, we estimate the width of this metastable state as $\Gamma = \Delta m \sim m \sim r_c^{-1}$. The interpretation of the dynamics of our model in terms of metastable gravitons has been considered in detail in [8] (subsequent to our original manuscript), and the direct calculation of the width indeed gave $\Gamma \sim r_c^{-1}$. In its own reference frame, the graviton disappears into the fifth dimension with time scale $\tau \sim \Gamma^{-1} \sim r_c$. This time scale determines the distance at which four-dimensional gravity of *static* sources gets modified. On the other hand, when the graviton moves in four-dimensional space-time with momentum $p \sim \omega$, there is an additional gamma factor of order $\gamma \sim \omega/m \sim \omega r_c$. The graviton therefore remains effectively four dimensional during a time of order $\gamma\tau \sim \omega r_c^2$ due to the relativistic time delay.

Because of this property, the leaking of the gravity waves into the extra dimension is negligible at relatively short wavelengths. However, the corresponding time scale becomes of order r_c for wavelengths of the same order. This is another manifestation of our observation that extra dimensions open up at the length scale r_c.

Finally, we point out that in more complicated higher-dimensional models, the long-distance properties of gravitational interactions may be even more intriguing. For example, if we modify our model by compactifying the fifth dimension at very large radius, $z_* \gg z_c$, the mass spectrum of Kaluza-Klein gravitons becomes discrete and the graviton zero mode reappears. However, the spacing between the masses may be tiny, depending on z_*. With appropriately chosen z_*, the five-dimensional gravity law, Eq. (19), will itself be valid in a finite interval of distances, and the four-dimensional gravity will again be restored at largest scales, well above r_c.

We conclude that higher-dimensional theories provide, somewhat unexpectedly, valid alternatives to four-dimensional gravity at *large distances*. With hindsight, it is perhaps not surprising that this happens, since, at the very large scale, the anti–de Sitter sandwich becomes very slim, and space-time is nearly flat; however, from the perspective of our putative Universe—the four-dimensional central wall—this conclusion is not as transparent. It would be worthwhile to explore such models in the context of cosmology and astrophysics. The long-distance phenomena in our Universe may become a window to microscopic extra dimensions.

We would like to thank Victor Berezin, Sergei Dubovsky, Dmitry Gorbunov, Maxim Libanov, and Sergei Troitsky for useful discussions. We would also like to thank Ian Kogan for drawing our attention to Ref. [3]. R. G. and V. A. R. acknowledge the hospitality of the Isaac Newton Institute for Mathematical Sciences, where this work was begun. R. G. was supported in part by the Royal Society, and V. A. R. and S. M. S. by the Russian Foundation for Basic Research, Grant No. 990218410.

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999), hep-th/9905221; Phys. Rev. Lett. **83**, 4690 (1999), hep-th/9906064.
[2] C. Charmousis, R. Gregory, and V. A. Rubakov, hep-th/9912160.
[3] I. I. Kogan, S. Mouslopoulos, A. Papazoglou, G. G. Ross, and J. Santiago, hep-ph/9912552.
[4] C. Csaki, M. Graesser, L. Randall, and J. Terning, hep-ph/9911406.
[5] W. D. Goldberger and M. B. Wise, hep-ph/9911457.
[6] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. **83**, 4922 (1999), hep-ph/9907447.
[7] J. Garriga and T. Tanaka, hep-th/9911055.
[8] C. Csaki, J. Erlich, and T. Hollowood, following Letter, Phys. Rev. Lett. **84**, 5932 (2000); G. Dvali, G. Gabadadze, and M. Porrati, hep-th/0002190.