Distillation of Greenberger-Horne-Zeilinger States by Selective Information Manipulation

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Methods for distilling Greenberger-Horne-Zeilinger (GHZ) states from arbitrary entangled tripartite pure states are described. These techniques work for virtually any input state. Each technique has two stages which we call primary and secondary distillations. Primary distillation produces a GHZ state with some probability, so that when applied to an ensemble of systems a certain percentage is discarded. Secondary distillation produces further GHZs from the discarded systems. These protocols are developed with the help of an approach to quantum information theory based on *absolutely selective information*, which has other potential applications.

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In the rapidly developing field of quantum information, it is possible to identify two main lines of investigation. On the one hand, it addresses basic questions on the fundamental nature of information, how it is embodied in quantum systems, how it can be quantified, and the extent to which physical properties can be reduced to informational ones [1]. On the other hand, it addresses specific operational issues, for example, how quantum information can be manipulated for applications such as quantum computation and teleportation [2].

In this Letter we try to bring together these two strands by proposing a new approach to the analysis of quantum information at a fundamental level, which leads directly to an operational technique for distilling maximally entangled tripartite [Greenberger-Horne-Zeilinger (GHZ)] states [3], using local operations and classical communication. The three qubits of the system are assumed to be physically separated and held by Alice, Bob, and Cara, respectively. [Throughout this Letter we use the term "maximally entangled" to refer to N-partite states that are N orthogonal; i.e., if the subsystems are two dimensional such a state would be $\sqrt{1/2}\,(|0000\ldots\rangle - |1111\ldots\rangle)$.]

Central to our approach is the notion of *absolutely selective* information, which has a straightforward interpretation in terms of classical information but can be seen as a basic distinguishing feature between quantum systems and their classical counterparts. We apply our approach to the specific problem of distilling GHZ states from arbitrary entangled tripartite pure states. We show that it is possible to distill, with a certain probability, a GHZ state from virtually any entangled tripartite pure state while retaining all three subsystems of the input state. As far as we know this is the first protocol of this type to be suggested. Our initial yield of GHZ states is then supplemented by an additional yield which involves sacrificing some subsystems. In this Letter we outline our approach and summarize our results. A more detailed exposition of the underlying analysis will be presented elsewhere [4].

The distinction between "selective" and "structural" information was addressed by Mackay [5] in the early days of classical information theory. Whereas structural information measures are based on an analysis of the *form* of possible events, selective information refers to *new* information gained from the occurrence of a specific event. For example, a signal might transmit a bit as one of two different wave forms; the selective information would be one bit, while the structural information, sufficient to describe all possible wave-form measurements, would be considerably more. *Absolutely* selective information signifies data that are irreducibly unpredictable, and hence genuinely new, in the sense that their unpredictability cannot be explained by the observer's ignorance. This type of information can arise only in a theory that is fundamentally stochastic, hence it is commonplace in quantum physics, but absent from classical physics. For a quantum state, the minimum local absolutely selective information (the minimum information generated by measuring one of the subsystems with a free choice of measurement basis) is exactly the same as the local entropy.

When considered as a quantitative measure, selective information is closely related to fundamental measures in quantum information theory. For example, the standard measure of entanglement for bipartite pure states [6] is numerically equal to the minimum local absolutely selective information. In a similar way, minimizing the absolutely selective information can be used to develop measures of nonorthogonality for quantum states [7]. In this Letter we show that absolutely selective information can be manipulated by an appropriate measurement procedure and apply this to an operational problem.

The problem we address is to transform a state $|\psi_{123}\rangle$,

$$|\psi_{123}\rangle = a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle + e|100\rangle + f|101\rangle + g|110\rangle + h|111\rangle, \quad (1)$$

into the state $|\psi_{\mathrm{GHZ}}\rangle = \sqrt{1/2}\,(|000\rangle - |111\rangle)$, with some probability. Let us first consider the minimum absolutely selective information A_i associated with each of the three qubits in $|\psi_{123}\rangle$:

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$$A_i = -[p_i \log_2 p_i + (1 - p_i)\log_2(1 - p_i)] \qquad i = 1, 2, 3, \tag{2}$$

where p_i and $1 - p_i$ are the eigenvalues of the reduced density operator ρ_i that describes system i when the other two subsystems are traced out,

$$\rho_i = \mathrm{Tr}_{jk}(|\psi_{ijk}\rangle\langle\psi_{ijk}|). \tag{3}$$

(Without loss of generality we adopt the convention that $p_i \geq 1/2$.)

It can be shown [8] that the *maximal* value of $\sum_{i=1}^{3} A_i$ for any tripartite pure state occurs uniquely for the GHZ state (or any local unitary transform of it), for which each of the p_i's is equal to $1/2$ and $\sum_{i=1}^{3} A_i = 3$. Our distillation procedure makes use of this fact by decreasing each of the p_i's in turn (with some probability), applying the procedure repeatedly until all of the p_i's are within some tolerance of $1/2$, at which point a GHZ state will necessarily have been distilled (up to a local unitary transformation).

The technique for decreasing the p_i's is similar to Bernstein and Bennett's *Procrustean* method [6]. We perform a positive operator valued (POV) measurement consecutively on each subsystem. To see how this works, let us carry out the procedure on subsystem 1. If we consider subsystems 2 and 3 as a single composite system, we can write the tripartite state as a Schmidt decomposition with respect to subsystem 1 and the composite 2-3 system:

$$|\psi_{123}\rangle = \sqrt{p_1}\,|+\rangle_1|\phi^+\rangle_{23} + \sqrt{1 - p_1}\,|-\rangle_1|\phi^-\rangle_{23}. \tag{4}$$

The minimum absolutely selective information for subsystem 1 is then given by

$$A_1 = -[p_1 \log_2 p_1 + (1 - p_1)\log_2(1 - p_1)]. \tag{5}$$

By carrying out an appropriate POV measurement on this subsystem, we can with some probability bring A_1 to its maximal value of 1. We introduce an ancilla qubit "a," which interacts unitarily with subsystem 1:

$$\begin{aligned}
&|+\rangle_1|0\rangle_a \to \alpha|+\rangle_1|0\rangle_a + \sqrt{1 - \alpha^2}\,|+\rangle_1|1\rangle_a, \\
&|-\rangle_1|0\rangle_a \to |-\rangle_1|0\rangle_a, \\
&|+\rangle_1|1\rangle_a \to \sqrt{1 - \alpha^2}\,|+\rangle_1|0\rangle_a - \alpha|+\rangle_1|1\rangle_a, \\
&|-\rangle_1|1\rangle_a \to |-\rangle_1|1\rangle_a.
\end{aligned} \tag{6}$$

We then measure the state of the ancilla. If we set $\alpha = \sqrt{(1 - p_1)/p_1}$ and the starting state of the ancilla to be $|0\rangle_a$, we will with probability $2(1 - p_1)$ find the ancilla in state $|0\rangle_a$, which projects the system into the state $\sqrt{1/2}\,(|+\rangle_1|\phi^+\rangle_{23} + |-\rangle_1|\phi^-\rangle_{23})$, for which $A_1 = 1$. With probability $2p_1 - 1$ we will measure state $|1\rangle_a$, in which case the procedure fails.

In Bernstein and Bennett's Procrustean technique, this one step followed by a local unitary transformation suffices to distill EPR pairs from arbitrary entangled bipartite states [6]. In the tripartite case, we then repeat the procedure on subsystems 2 and 3, which projects the system into states for which $p_2 = 1/2, A_2 = 1$ and $p_3 = 1/2, A_3 = 1$, respectively. However, if we simply carry out a single POV measurement of this type on each of the three subsystems in turn, the resulting tripartite state will not in general be a GHZ state. Each step of the process is nonunitary, since the tripartite system can be discarded at each stage if the wrong result for the POV measurement is obtained. While the A_i are conserved by unitary operations on the local subsystems, they are not conserved in general for nonunitary operations. Hence, when we carry out a POV measurement on bit i to project the system into a state for which $p_i = 1/2$ and $A_i = 1$, this will disrupt the values of p and A for the other two qubits.

Nevertheless, it transpires that, for most tripartite states, repeated application of this type of POV measurement will steadily move the input state towards a GHZ state until it gets arbitrarily close to it. (Exceptions will be identified later.) There are a number of plausible ways to measure "closeness" to a GHZ state. Three such measures are

$$D_p \equiv \sum_{i=1}^{3} p_i - 3/2, \tag{7a}$$

$$D_S \equiv 3 - \sum_{i=1}^{3} A_i, \tag{7b}$$

$$D_2 \equiv 3/4 - \sum_{i=1}^{3} p_i(1 - p_i). \tag{7c}$$

We introduce this last quantity because it is more tractable analytically than D_p and D_S, being a simple polynomial function of the coefficients of $|\psi\rangle_{123}$.

We simulated this process on a large sample of randomly chosen initial states, treating the real and imaginary parts of the coefficients of (1) as random variables uniformly distributed on the surface of a sixteen-dimensional hypersphere. Numerical analysis of this sample shows that, for a large fraction of these states, D_p approaches zero to an accuracy of 10^{-3} after just two complete iterations (i.e., two POV measurements performed on each of the three subsystems), while virtually all do so within four iterations. Interestingly, we find that, in every case examined (aside from the exceptions given below), D_p decreases monotonically toward zero with each step of the procedure, whereas D_S and D_2 can fluctuate, though of course their general trend is downward.

The results presented so far are supported by numerical analysis only. However, there is a closely related procedure for which we have derived analytical proof of efficacy for virtually any input state [4]. In this second method, instead of reducing each probability p_i to $1/2$ in turn using POV measurements, the probabilities are reduced by a small amount ϵ, so that with each step the state changes infinitesimally in the limit $\epsilon \to 0$. The proof follows fairly straightforwardly by deriving the changes in the state coefficients from the procedure, then using these to derive the change in D_2. By changing to the Schmidt basis for all three bits—a very useful standard form—one can, with some effort, show that $p_i(1 - p_i)$ can never decrease for

$i = 1, 2, 3$, and will only remain unchanged for certain very special initial states detailed below. This monotonicity implies monotonicity for D_2, D_p, and D_S, so that all of these quantities diminish steadily as the state approaches a GHZ state. This infinitesimal method would be quite challenging experimentally, but it is analytically interesting due to its relative tractability.

The protocols described thus far correspond to what we call "primary" distillation. They will give a specific yield (i.e., surviving percentage) of GHZ states if a collection of systems is supplied in a given input state. This yield can be straightforwardly calculated for the large-step procedure; after each POV measurement on the ith subsystem a proportion $2(1 - p_i)$ of the systems is retained. The yield of GHZ states for the primary distillation process averaged approximately 9.2% for the evenly distributed sample of input states we analyzed, but this will clearly depend strongly on the initial distribution.

Average yields for the infinitesimal procedure were 9.8%. The chance of the procedure failing on any given step is quite small, but over many steps the number of failures mounts. The difference between the infinitesimal and big-step procedure is interesting when contrasted with the bipartite Procrustean technique. In the bipartite case, there is no advantage to using small steps over a single large step; the yields are the same in both cases. Clearly in the more elaborate tripartite procedure there is a difference.

This yield can be greatly enhanced by a process of *secondary distillation*, which makes use of those systems discarded during primary distillation. When we carry out the initial POV measurement on subsystem 1 for the input state $|\psi\rangle_{123}$ given by Eq. (1), with probability $2p_1 - 1$, we fail to obtain the desired result. However, this failure will leave the discarded system in the state $|+\rangle_1|\phi^+\rangle_{23}$, where $|\phi^+\rangle_{23}$ is in general an entangled bipartite state of subsystems 2 and 3. Similarly, failures at later steps of the primary distillation process can yield entangled bipartite states of subsystems 1 and 2 and of subsystems 1 and 3. Thus, when the primary distillation procedure has been completed on a collection of systems in a given input state, we will have an additional residue of entangled bipartite states of subsystems 1 and 2, 1 and 3, and 2 and 3. These entangled pairs can be distilled to EPR pairs by standard techniques [6], and the resulting EPR pairs can be used to prepare further GHZ triplets. (For example, if Alice shares one EPR pair with Bob and another with Cara, she can distribute a GHZ state by preparing it locally and then teleporting the states of two of the subsystems to Bob and Cara with the help of the two EPR pairs.) This is quite similar to the method of [9].

If, when primary distillation is completed, we produce N_{23} EPR pairs of subsystems 2 and 3, N_{31} EPR pairs of subsystems 3 and 1, and N_{12} EPR pairs of subsystems 1 and 2 from the discarded systems, we will be able to distill further $(N_{23} + N_{31} + N_{12})/2$ GHZ triplets (in the case where none of the Ns is greater than the sum of the other two) or $(N_{jk} + N_{ki})$ GHZ triplets (if $N_{ij} > N_{jk} + N_{ki}$). Numerical analysis indicates that the average yield for secondary distillation of GHZ states, for the random sample considered, is approximately 27.5% for the large-step procedure, giving a total yield of about 36.7%. The infinitesimal technique does even better, giving a secondary yield of 29.4% for a total yield of 39.2%.

Since the bulk of this yield comes from the production of EPR pairs, one might reasonably ask how these methods compare to simply producing EPR pairs (with no primary distillation) and then using these pairs to produce GHZ triplets directly [9]. EPR pairs are produced by measuring one of the subsystems in such a way as to maximize the pairwise entanglement between the other two bits and then distilling perfect EPR pairs from the resulting states. For the same random sample of states, this technique produces an average yield of 31.5%, lower than either of the other two techniques and not much higher than the secondary yield alone. This does not, of course, prove that it is worse for every initial state. However, the closer the initial state is to a GHZ [using any of our distance measures (7a)–(7c)], the better the distillation procedures presented in this paper perform, while producing GHZs from EPR pairs has a maximum yield of 50%.

For some special cases these protocols will not work as described. If the original input state is not three-party entangled, the protocol will fail completely; that is, if the original state can be written as $|\chi\rangle_i|\zeta\rangle_{jk}$, no three-party entanglement will be distillable by either primary or secondary distillation. There is another set of states for which primary distillation fails, but which can still produce GHZ states by secondary distillation. This set consists of tripartite input states with just three components, where each component is biorthogonal (but not triorthogonal) to the other two, and local unitary transforms of such states. For example, the state $|\psi\rangle_{\rm tr} = b|001\rangle + c|010\rangle + e|100\rangle$ is of this type. We call such states "triple" states; all have $D_p \geq 1/2$, though a substantial subset has $D_p = 1/2$ exactly. Both forms of the primary distillation process take triple states to triple states, so that the GHZ state will never be produced. The large-step procedure causes all triple states to converge to the state $|\psi_{\rm GM}\rangle = \sqrt{1/2}|001\rangle + \sqrt{(\sqrt{5} - 1)/4}|010\rangle + \sqrt{(3 - \sqrt{5})/4}|100\rangle$, at which point any further steps will simply result in a cyclic shuffling of the component amplitudes. We call this attractor state the "golden mean" state because of the appearance of the golden mean in the amplitudes. The infinitesimal procedure leaves all triple states with $D_p = 1/2$ unchanged. This procedure can also cause certain other states with $D_p > 1/2$ to converge to triple states rather than GHZ states, though most do not. The large-step procedure may also take some states to triple states. Triple states also play an interesting role in the work of Coffman, Kundu, and Wootters [10], where they minimize the *residual three-tangle*, and of Carteret and Sudbery [11], who showed that they behave atypically under local unitary transformations.

Although triple states do not yield any GHZ states by primary distillation, they can of course produce them by secondary distillation. Moreover, it is possible to move off of a triple state (with some probability) by performing a POV measurement in a basis other than the Schmidt basis. That is, instead of using the basis $|+\rangle_1$ and $|-\rangle_1$ in the transformation (6), one uses a different basis, such as $(|+\rangle_1 \pm |-\rangle_1)/\sqrt{2}$. Setting α to a reasonable value (such as $\alpha = \sqrt{1/2}$) will then take triple states to nontriple states. However, the new state produced by this procedure will in general reconverge to a triple state.

We have shown that manipulation of absolutely selective information can be used to distill maximally entangled tripartite states from arbitrary tripartite entangled pure states. This method will not work for systems with four or more subsystems, since, in these, $p_i = 1/2$ does not uniquely determine the maximally entangled state. It may be that related techniques might succeed, however, if it is possible to manipulate other locally unitarily invariant parameters by local POVs and classical communication.

Even in the three-qubit case, however, the procedure we have described is surely not optimal. We can see this by considering states of the form $|\psi\rangle = a|000\rangle + b|111\rangle$ with a and b real and $a^2 + b^2 = 1$. These *generalized GHZ states* give a yield of $2(1 - a^2)$ (assuming $a^2 > b^2$) by our technique, both for large and infinitesimal step sizes. The asymptotic algorithms for distilling bipartite entanglement also work for generalized GHZs in the tripartite case, giving a yield of $-(a^2 \log a^2 + b^2 \log b^2) \geq 2(1 - a^2)$. For this case, therefore, our algorithm is clearly suboptimal.

The optimal distillation technique for a general three-qubit state is not known [12], but would almost certainly make use of joint manipulations on many copies of the input state. Nor is an asymptotically *reversible* distillation technique known for GHZ states [13]. It would be interesting to compare the yields of these two hypothetical techniques. In the bipartite case they are the same, but this need not be so in the tripartite case. Indeed, if the reversible GHZ distillation technique produced any extra two-party entanglement, one would generally expect to be able to produce further GHZ states by an irreversible secondary distillation stage. This suggests that the algorithm giving the optimal yield of GHZs will probably not be reversible. It would also be interesting to compare our yield of GHZ states to some standard measure of tripartite entanglement. Lacking such a measure, however, the best that can be done is to compare different distillation techniques to each other.

There may be a number of other problems in quantum information theory which are amenable to an approach focusing on the absolutely selective information content of quantum systems. For example, work in progress suggests than such an approach can be useful in the analysis of nonorthogonality. Since selective information is a classical concept, this approach also provides a valuable link between classical and quantum information.

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