New Variational Principle for the Vlasov-Maxwell Equations

Alain J. Brizard

Lawrence Berkeley National Laboratory, University of California, Berkeley, California 94720

(Received 28 February 2000)

A new Eulerian variational principle is presented for the Vlasov-Maxwell equations. This principle uses constrained variations for the Vlasov distribution in eight-dimensional extended phase space. The standard energy-momentum conservation law is then derived explicitly by the Noether method. This new variational principle can be applied to various reduced Vlasov-Maxwell equations in which fast time scales have been asymptotically eliminated (e.g., low-frequency gyrokinetic theory).

PACS numbers: 52.25.Dg, 03.50.De

The variational formulation for the Vlasov-Maxwell equations has been a topic of interest in plasma physics ever since Low presented his *Lagrangian* variational principle [1], which combined the standard action functional for the electromagnetic field with an action functional for a Vlasov distribution of particles. Since then a variety of variational formulations for the Vlasov-Maxwell equations have appeared [2–4] (see Refs. [5] and [6] for recent works on this topic). A *Eulerian* variational principle for the Vlasov-Maxwell equations was recently derived by Cendra *et al.* [6] by transforming the Low Lagrangian formulation and introducing *constrained* variations on the phase-space Hamiltonian vector field $\mathbf{u} \equiv \dot{\mathbf{z}}$ and on the phase-space Vlasov density F in terms of a virtual displacement $\mathbf{w}$ in six-dimensional phase space: $\delta\mathbf{u} = (\partial_t + \mathbf{u}\cdot\partial_{\mathbf{z}})\mathbf{w} - \mathbf{w}\cdot\partial_{\mathbf{z}}\mathbf{u}$ and $\delta F = -\partial_{\mathbf{z}}\cdot(F\mathbf{w})$. These expressions are in complete analogy with the Eulerian variations for the fluid velocity $\mathbf{v}$ and fluid density n for ideal fluids expressed in terms of the virtual fluid displacement $\boldsymbol{\xi}$ [7], where $\delta\mathbf{v} = (\partial_t + \mathbf{v}\cdot\nabla)\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{v}$ and $\delta n = -\nabla\cdot(n\boldsymbol{\xi})$. The expression for $\delta\mathbf{v}$ is known as the Lin constraints on the variation of the fluid velocity; the general framework for constrained variations in continuum theories is discussed in Refs. [6] and [8].

The purpose of this Letter is to present a new Eulerian variational principle for the Vlasov-Maxwell equations which is simpler than all previous variational formulations. In particular, whereas Cendra *et al.* [6] consider constrained variations on the particle dynamics and the Vlasov distribution expressed in terms of a six-dimensional virtual displacement vector field $\mathbf{w}$, our new variational principle considers constrained variations of the Vlasov distribution on eight-dimensional extended phase space expressed in terms of the canonical Poisson bracket and a single scalar field δS which generates a virtual displacement $Z^a \to Z^a + \delta Z^a$ in extended phase space, with $\delta Z^a \equiv \{Z^a, \delta S\}_Z$.

The new variational principle for the Vlasov-Maxwell equations is written in terms of the following action functional:

$$\mathcal{A}[\mathcal{F}, A_\mu] \equiv \int d^8Z\, \mathcal{F}(Z)[w - H(\mathbf{x},\mathbf{p},t)] + \int d^4x \frac{1}{16\pi} \mathsf{F}_{\mu\nu}\mathsf{F}^{\nu\mu}, \tag{1}$$

where $Z \equiv (\mathbf{x},\mathbf{p}; w, t)$ denotes canonical coordinates on eight-dimensional extended phase space (here, $\mathbf{x}$, $\mathbf{p}$, and w denote a particle's position, canonical momentum, and energy, respectively), $\mathcal{F}(Z)$ denotes the Vlasov distribution on this extended phase space, and $H(\mathbf{x},\mathbf{p},t)$ denotes the time-dependent single-particle Hamiltonian on regular six-dimensional phase space. The second term in (1) is the standard action functional for the electromagnetic field, with the field tensor defined in terms of the four-potential $A_\mu = (\phi, \mathbf{A})$ as $\mathsf{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$; here and henceforth, we use the Minkowski space-time metric $g^{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ and the covariant notation $\partial_\mu = (c^{-1}\partial_t, \nabla)$. The variational variables $(\mathcal{F}, A_\mu)$ in (1) are said to be *Eulerian* since their variations are evaluated at a fixed point Z in extended phase space for $\delta\mathcal{F}(Z)$ or at a fixed point $x = (ct, \mathbf{x})$ in four-dimensional space-time for $\delta A^\mu(x)$.

For a charged-particle species with mass m and charge q, the (nonrelativistic) single-particle Hamiltonian is $H(\mathbf{z},t) \equiv |\mathbf{p} - (q/c)\mathbf{A}(\mathbf{x},t)|^2/2m + q\phi(\mathbf{x},t)$, where $\mathbf{z} = (\mathbf{x},\mathbf{p})$ denotes canonical coordinates in six-dimensional phase space. We note that the dynamics of a charged particle in eight-dimensional extended phase space is constrained to take place on the energy surface

$$w - H(\mathbf{z},t) = 0. \tag{2}$$

Thus the *physical* Vlasov distribution $\mathcal{F}(Z)$ must be of the form

$$\mathcal{F}(Z) \equiv \delta[w - H(\mathbf{z},t)]\, f(\mathbf{z},t), \tag{3}$$

where $f(\mathbf{z},t)$ is the time-dependent Vlasov distribution on six-dimensional phase space. The energy constraint (2) is easily obtained from the action functional (1) if we consider arbitrary variations in $\mathcal{F}$; i.e., $\delta\mathcal{A}/\delta\mathcal{F} = 0$ implies

 0031-9007/00/84(25)/5768(4)$15.00

(2). The Vlasov equation is obtained by considering the following constrained variation of $\mathcal{F}$:

$$\delta\mathcal{F} \equiv \{\delta S, \mathcal{F}\}_Z \equiv (\partial_\mu \delta S)(D^\mu \mathcal{F}) - (D^\mu \delta S)(\partial_\mu \mathcal{F}), \tag{4}$$

where we use the covariant notation $p_\mu \equiv (-w/c, \mathbf{p})$ and $D^\mu \equiv (-c\partial_w, \partial_{\mathbf{p}})$, while δS is the generating scalar field for an infinitesimal (virtual) displacement $Z \to Z + \delta Z$ in extended phase space, with $\delta Z^a \equiv \{Z^a, \delta S\}_Z$ for each eight-dimensional phase-space coordinate $(a = 1, \ldots, 8)$. This form for $\delta\mathcal{F}$ ensures that all of the Casimir invariants $C_\Phi \equiv \int d^8 Z\, \Phi(\mathcal{F})$ (e.g., entropy) associated with the Vlasov equation are preserved by $\delta\mathcal{F}$ since $\delta C_\Phi \equiv \int d^8 Z \{\delta S, \Phi(\mathcal{F})\}_Z = 0$. We note that, in analogy with the previous Eulerian variational principle [6], Eq. (4) can also be written as a phase-space divergence $\delta\mathcal{F} = -\partial_a(\delta Z^a \mathcal{F})$.

The variational principle for the Vlasov-Maxwell equations can be expressed as

$$\delta\mathcal{A} \equiv \int d^4x\, \delta\mathcal{L} = 0, \tag{5}$$

where the four-dimensional Lagrangian density is

$$\mathcal{L} \equiv \frac{1}{16\pi} \mathsf{F}_{\mu\nu}\mathsf{F}^{\nu\mu} + \int d^4p\, \mathcal{F}(Z)(w - H). \tag{6}$$

Upon variation, we obtain, from (6),

$$\delta\mathcal{L} \equiv \int d^4p\, \delta S\{\mathcal{F}, (w - H)\}_Z + \delta A_\mu\left[\frac{1}{4\pi}\partial_\nu \mathsf{F}^{\nu\mu} - \frac{q}{c}\int d^4p\, \mathcal{F}(Z) D^\mu(w - H)\right] + \partial_\nu\left[\frac{1}{4\pi}\delta A_\mu \mathsf{F}^{\mu\nu} + \int d^4p\, \delta S(w - H) D^\nu \mathcal{F}(Z)\right]. \tag{7}$$

Since the last term is an exact divergence in four-dimensional space-time, it does not contribute to the variational principle (5). For arbitrary Eulerian variations δS and δA_μ, we obtain, respectively, the Vlasov equation in eight-dimensional phase space,

$$\{\mathcal{F}, (w - H)\}_Z = 0, \tag{8}$$

and the Maxwell equations,

$$\partial_\nu \mathsf{F}^{\nu\mu} = \frac{4\pi q}{c}\int d^4p\, \mathcal{F}(Z) D^\mu(w - H). \tag{9}$$

Substituting the physical representation (3) for $\mathcal{F}$ into (8), we obtain the regular Vlasov equation in six-dimensional phase space:

$$0 = \{f, (H - w)\}_Z \equiv \frac{\partial f}{\partial t} + \{f, H\}_{\mathbf{z}}, \tag{10}$$

where $\{,\}_{\mathbf{z}}$ denotes the canonical Poisson bracket in six-dimensional phase space. Substituting $D^\mu(w - H) = -(c, \mathbf{v}) \equiv -v^\mu$ into (9), where $\mathbf{v} \equiv (\mathbf{p} - q\mathbf{A}/c)/m$ is the particle's velocity, the Maxwell equations (9) become

$$\partial_\nu \mathsf{F}^{\nu\mu}(x) = -\frac{4\pi q}{c}\int d^3p\, v^\mu f(\mathbf{x}, \mathbf{p}, t). \tag{11}$$

The remaining Maxwell equations are expressed in terms of the identity $\partial^\alpha \mathsf{F}^{\mu\nu} + \partial^\mu \mathsf{F}^{\nu\alpha} + \partial^\nu \mathsf{F}^{\alpha\mu} = 0$. The derivation of the Vlasov-Maxwell equations (8) and (9) [or (10) and (11)] from the Eulerian variational principle (5) is far simpler than previous derivations based on other variational principles.

We now show how to use this new variational principle to obtain the energy-momentum conservation law for the Vlasov-Maxwell equations. For this purpose, we consider

$$\delta\mathcal{L} \equiv \partial_\nu\left[\frac{1}{4\pi}\delta A_\mu \mathsf{F}^{\mu\nu} + \int d^4p\, \delta S(w - H) D^\nu \mathcal{F}(Z)\right], \tag{12}$$

obtained from (7) after the dynamical equations (8) and (9) have been substituted. Until now, the variations $(\delta S, \delta A_\mu)$ have been treated as independent and arbitrary. Under an infinitesimal space-time translation $x \to x + \delta x$, with $\delta x \equiv (c\delta t, \delta\mathbf{x})$, the expressions for $(\delta S, \delta A_\mu)$ and $\delta\mathcal{L}$ are, respectively, $\delta S \equiv p_\mu \delta x^\mu$, $\delta A_\mu \equiv \mathsf{F}_{\mu\nu}\delta x^\nu - \partial_\mu(A_\nu \delta x^\nu)$, and $\delta\mathcal{L} \equiv -\partial_\mu(\delta x^\mu \mathcal{L})$. The first expression implies that δS is the generating scalar field for an infinitesimal space-time translation in extended phase space (i.e., $\delta x^\mu \equiv \{x^\mu, \delta S\}_Z$). The second expression can be written as $\delta A_\mu dx^\mu \equiv -\mathsf{L}_{\delta x}(A_\mu dx^\mu)$, where $\mathsf{L}_{\delta x}$ is the Lie derivative along δx. The last expression can be expressed as $\delta\mathcal{L}\,\Omega \equiv -\mathsf{L}_{\delta x}(\mathcal{L}\,\Omega)$, where Ω is the oriented space-time volume element.

For a constant space-time translation δx^μ, the Noether equation (12) becomes the energy-momentum conservation law for the Vlasov-Maxwell equations

$$0 = \partial_\mu\left[g^{\mu\nu}\mathcal{L} - \frac{1}{4\pi}(\mathsf{F}^\mu_{\ \alpha}\mathsf{F}^{\alpha\nu} - \mathsf{F}^{\mu\alpha}\partial_\alpha A^\nu) + \int d^4p\, p^\nu(w - H) D^\mu \mathcal{F}\right]. \tag{13}$$

Here, $\mathcal{L} \equiv \mathsf{F}_{\mu\nu}\mathsf{F}^{\nu\mu}/16\pi$ since the Vlasov part in (6) vanishes identically when the physical representation (3) is used for $\mathcal{F}(Z)$. Next, we note that, owing to the antisymmetry of $\mathsf{F}^{\mu\nu}$, the third term in (13) can be rewritten as

$$\partial_\mu\left(\frac{\mathsf{F}^{\mu\alpha}}{4\pi}\,\partial_\alpha A^\nu\right) = \partial_\mu\left(\frac{A^\nu}{4\pi}\,\partial_\alpha \mathsf{F}^{\alpha\mu}\right) = \partial_\mu\left[\frac{q}{c}\int d^4p\, A^\nu D^\mu(w - H)\mathcal{F}(Z)\right], \tag{14}$$

where the Maxwell equations (11) were used to obtain the last equality in (14). Combining these terms and integrating by parts in the Vlasov part in (13) yield the energy-momentum conservation law

$$\partial_\mu \mathsf{T}^{\mu\nu} \equiv 0\,, \tag{15}$$

where the energy-momentum tensor for the Vlasov-Maxwell equations is

$$\mathsf{T}^{\mu\nu} \equiv \frac{1}{4\pi}\left(\frac{g^{\mu\nu}}{4}\,\mathsf{F}_{\alpha\beta}\mathsf{F}^{\beta\alpha} - \mathsf{F}^{\mu}_{\alpha}\mathsf{F}^{\alpha\nu}\right) + \int d^4p\,\mathsf{K}^{\mu\nu}\mathcal{F}(Z)\,, \tag{16}$$

with $\mathsf{K}^{\mu\nu} \equiv -(p^\nu - qA^\nu/c)D^\mu(w - H)$, i.e., $\mathsf{K}^{\mu 0} = (w - q\phi)(1, \mathbf{v}/c)$ and $\mathsf{K}^{\mu i} = (mcv^i, mv^i\mathbf{v})$. Inserting these expressions into (16) and performing the necessary w integration over $\delta(w - H)$ yield the standard energy-momentum conservation law for the Vlasov-Maxwell equations in terms of the electromagnetic field $\mathsf{F}^{\mu\nu}$ and the Vlasov distribution $f(\mathbf{x}, \mathbf{p}, t)$ on six-dimensional phase space.

Having demonstrated that the action functional (1) leads to the correct variational principle (5) for the Vlasov-Maxwell equations (8) and (9) and that the energy-momentum conservation law (15) is properly derived by the Noether method, we now turn our attention to applications in the context of Hamiltonian perturbation theory. In particular, we look at the application of Hamiltonian Lie-perturbation techniques [9] to asymptotically eliminate fast degrees of freedom in Hamiltonian systems. For this purpose, we consider the Vlasov part of the action functional (1):

$$\mathcal{A}_V[\mathcal{F}] \equiv -\int d^8Z\,\mathcal{F}(Z)\mathcal{H}(Z)\,, \tag{17}$$

where $\mathcal{H}(Z) \equiv H(\mathbf{z}, t) - w$ is the extended phase-space Hamiltonian. In Hamiltonian Lie-perturbation theory [9], the asymptotic elimination of a fast degree of freedom (represented by an angle variable θ) proceeds by a near-identity canonical phase-space transformation $Z \to \overline{Z}(Z, \epsilon) \equiv T_\epsilon Z$, where ϵ denotes a dimensionless perturbation parameter (i.e., $H = H_0 + \epsilon H_1 + \ldots$). Here, the near-identity canonical phase-space transformation is explicitly expressed in terms of generating scalar fields $(S_1, S_2, \ldots)$:

$$\overline{Z}(Z, \epsilon) = Z + \epsilon\{S_1, Z\}_Z + \epsilon^2\left(\{S_2, Z\}_Z + \frac{1}{2}\{S_1, \{S_1, Z\}_Z\}_Z\right) + \ldots\,, \tag{18}$$

where S_n is chosen to remove the fast time scale at order ϵ^n in the Hamiltonian dynamics. As a result, an action variable $\overline{J}$ (an adiabatic invariant) is constructed as an asymptotic expansion in powers of ϵ with the following property: if the perturbation analysis is performed up to ϵ^n, then $d\overline{J}/dt \equiv -\partial\overline{H}/\partial\overline{\theta} = \mathcal{O}(\epsilon^{n+1})$.

Under the near-identity canonical phase-space transformation $Z \to \overline{Z}(Z, \epsilon) \equiv T_\epsilon Z$, the Vlasov action functional (17) becomes

$$\mathcal{A}_V[\overline{\mathcal{F}}] \equiv -\int d^8\overline{Z}\,\overline{\mathcal{F}}(\overline{Z})\overline{\mathcal{H}}(\overline{Z})\,, \tag{19}$$

where $\overline{\mathcal{F}}(\overline{Z}) \equiv (T_\epsilon^*)^{-1}\mathcal{F}(\overline{Z})$ denotes the new Vlasov distribution expressed as the *pull back* of the old Vlasov distribution $\mathcal{F}$. Note that, by construction, we have $\overline{\mathcal{H}}(\overline{Z}) \equiv (T_\epsilon^*)^{-1}\mathcal{H}(\overline{Z}) = \overline{\mathcal{H}}_0 + \epsilon\overline{\mathcal{H}}_1 + \epsilon^2\overline{\mathcal{H}}_2 + \ldots$, where $\overline{\mathcal{H}}_0 = \mathcal{H}_0(\overline{Z}) \equiv H_0(\overline{Z}) - \overline{w}$, $\overline{\mathcal{H}}_1 \equiv \langle H_1\rangle$, and $\overline{\mathcal{H}}_2 \equiv \langle H_2\rangle - (1/2)\langle\{S_1, H_1\}_Z\rangle$, where $\langle\,\rangle$ denotes a time average over the fast time scale. The new Hamiltonian $\overline{\mathcal{H}}$ in (19) is therefore independent of the fast time scale. By applying the variational principle (5) with the Vlasov part given by (19), it is a simple task to derive the *reduced* Vlasov equation $\{\overline{\mathcal{F}}, \overline{\mathcal{H}}\}_{\overline{Z}} = 0$.

Reduced Vlasov formulations play a fundamental role in understanding the self-consistent nonlinear (turbulent) dynamics of fusion, space, and astrophysical plasmas. As an explicit example, we now briefly consider the case of low-frequency nonlinear gyrokinetic Vlasov-Maxwell theory; the interested reader will find more details in Ref. [10]. One important result of the Eulerian variational formulation of low-frequency nonlinear gyrokinetic Vlasov-Maxwell theory is the derivation (by the Noether method) of a *local* gyrokinetic energy conservation law $\partial_t\mathcal{E} + \nabla\cdot\mathbf{S} = 0$, where the gyrokinetic energy density is

$$\mathcal{E}(\mathbf{x}, t) = \int d^6\overline{Z}\,\delta^3(\mathbf{x} - \overline{\mathbf{R}})\overline{f}(\overline{Z}, t) \times (\overline{H} - \epsilon q\langle(T_{gy}^*)^{-1}\phi_{1gc}\rangle) + \frac{\epsilon^2}{8\pi}(|\mathbf{E}_1|^2 + |\mathbf{B}_1|^2)\,, \tag{20}$$

while the gyrokinetic energy density flux is

$$\mathbf{S}(\mathbf{x},t) = \int d^6\overline{Z}\,\delta^3(\mathbf{x} - \overline{\mathbf{R}})\overline{f}(\overline{Z},t) \times \left(\overline{H}\,\dot{\overline{\mathbf{R}}} - \epsilon q\langle(T_{gy}^*)^{-1}\mathbf{v}\phi_{1gc}\rangle\right) + \epsilon\phi_1\left(\mathbf{J}_0 - \frac{\epsilon}{4\pi}\nabla\partial_\dagger\phi_1\right) + \frac{c\epsilon^2}{4\pi}\mathbf{E}_1 \times \mathbf{B}_1\,. \tag{21}$$

Here, $(\phi_1, \mathbf{A}_1)$ denote perturbation potentials, $\mathbf{E}_1 \equiv -\nabla\phi_1$ and $\mathbf{B}_1 \equiv \nabla \times \mathbf{A}_1$ denote the low-frequency perturbed electric and magnetic fields, $\overline{Z} \equiv (\overline{\mathbf{R}}, \overline{p}_\parallel, \overline{\mu}, \overline{\theta})$ denotes *gyrocenter* phase-space coordinates, $\overline{H}$ and $\overline{f}(\overline{Z},t)$ denote the nonlinear gyrocenter Hamiltonian and Vlasov distribution, respectively, $\dot{\overline{\mathbf{R}}} \equiv \{\overline{\mathbf{R}}, \overline{H}\}_{gy}$ denotes gyrocenter velocity ($\mathbf{v}$ denotes particle velocity), and $\mathbf{J}_0 \equiv (c/4\pi)\nabla \times \mathbf{B}_0$ denotes the unperturbed current density.. The gyrocenter pull back $(T_{gy}^*)^{-1}$ and Poisson bracket $\{,\}_{gy}$, as well as other definitions, can be found in Ref. [11].

In summary, a new Eulerian variational principle for the Vlasov-Maxwell equations is presented. Based on this Eulerian principle, the variational formulation of the Vlasov-Maxwell equations is simpler than previous formulations based on Lagrangian, mixed Lagrangian-Eulerian, or Eulerian principles. A great advantage of this new variational principle is how efficiently the Eulerian variational principles for reduced Vlasov-Maxwell equations can be obtained. Future applications of this work includes, for example, the variational formulation of low-frequency bounce-gyrokinetic Vlasov-Maxwell equations [12].

This work was supported by the U.S. Department of Energy under Contract No. DE-AC03-76SFOO098.

[1] F. E. Low, Proc. R. Soc. London A **248**, 282 (1958).
[2] P. A. Sturrock, Ann. Phys. (N.Y.) **4**, 306 (1958).
[3] J. J. Galloway and H. Kim, J. Plasma Phys. **6**, 53 (1971).
[4] R. L. Dewar, J. Plasma Phys. **7**, 267 (1972).
[5] H. Ye and P. J. Morrison, Phys. Fluids B **4**, 771 (1992).
[6] H. Cendra, D. D. Holm, M. J. W. Hoyle, and J. E. Marsden, J. Math. Phys. **39**, 3138 (1998).
[7] W. A. Newcomb, Nucl. Fusion (1962 Suppl., Part 2), 451 (1962).
[8] D. D. Holm, J. E. Marsden, and T. S. Ratiu, Adv. Math. **137**, 1 (1998).
[9] J. R. Cary and A. N. Kaufman, Phys. Fluids **24**, 1238 (1981).
[10] A. J. Brizard and A. N. Kaufman (to be published).
[11] A. Brizard, J. Plasma Phys. **41**, 541 (1989).
[12] A. J. Brizard (to be published).