Quantum State Tomography with Array Detectors

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I propose a method for measuring the quantum state of an optical field that occupies a mode having a complicated spatial structure. The technique uses array detectors and a single, plane-wave local oscillator beam. The advantage of using array detectors is that the local oscillator is *not* mode matched to the field being measured, yet the deleterious effects of this mismatch on the effective detection efficiency are greatly reduced compared to using single detectors. Indeed, when the spatial mode of the signal field is describable by a real function, the effective mode-matching efficiency is unity.

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Quantum state tomography (QST) is a well established technique for measuring the quantum mechanical state of a light mode [1–3]. Since its first experimental demonstration in Ref. [2], it has seen numerous theoretical and experimental augmentations (for reviews, see Refs. [4,5]). This work has included techniques for measuring multimode systems [6–9], and extensions to state measurements of molecular vibrations [10], trapped ions [11], and atomic beams [12].

The usual state-measurement technique involves interfering the signal field to be measured with a coherent-state local oscillator (LO) on a beam splitter [2,3]. Interference between the signal and LO fields is essential, and any mode mismatch (be it spatial or time/frequency) between the signal and the LO effectively acts as a loss of detection efficiency which degrades the measurement [3,4]. It is well known that loss of efficiency smooths the measured distributions in QST, an undesirable effect as it can mask nonclassical effects. For example, the squeezed mode emitted from an optical-parametric amplifier (OPA) pumped by a Gaussian beam is known to have a complicated spatial structure [13,14]. This fact makes it extremely difficult to mode match an LO to this squeezed field [15] and can severely reduce the detected squeezing.

Here I present a new technique that uses array detectors to measure the quantum state of an optical field. This technique is novel in that the use of array detectors can eliminate the effective losses incurred due to mode mismatch between the signal and local oscillator fields. In other words, the LO and signal fields are *not* mode matched, yet vacuum noise does not leak in and degrade the state measurement of the signal field. Experimentally this offers an advantage, because it means that one may use a plane-wave LO to measure a field with a complicated mode structure (such as the squeezed mode of an OPA) with no loss of efficiency due to mode mismatch.

Consider a narrow-band, electromagnetic field propagating in the z direction. At $z = 0$, in a plane perpendicular to the propagation direction, the operator for the positive-frequency part of the electric field can be expressed as

$$\hat{E}_A^{(+)}(x,y) = \left(\frac{2\pi\hbar\varpi}{L}\right)^{1/2} \sum_n \hat{a}_n u_n(x,y)\,, \qquad (1)$$

where ϖ is the mean frequency of the field, L is the longitudinal quantization length, and x and y are the transverse coordinates in this plane. The sum is over a complete set of independent modes, with $\hat{a}_n$ being the annihilation operator and $u_n(x,y)$ being the mode function corresponding to mode n. The mode functions form an orthonormal set of basis functions:

$$\int_{-D_x/2}^{D_x/2}\int_{-D_y/2}^{D_y/2} dx\,dy\, u_n^*(x,y) u_{n'}(x,y) = \delta_{n,n'}\,, \qquad (2)$$

where D_x and D_y are the quantization lengths (detector widths) in the x and y directions. As will be shown below, by using the measurement scheme described in this Letter, it is possible to select out a single mode from the sum in Eq. (1). This mode is referred to as the measured mode, and I denote it by $n = m$. This measured mode is described by the spatial function $u_m(x,y)$, and all of the other mode functions in Eq. (1) can be obtained from it via the standard Schmidt procedure [16].

This field is to be detected with an array detector, located in the plane $z = 0$ where the field is quantized. The detector consists of a two dimensional $N \times N'$ array of adjacent photodetectors (pixels). Each pixel has an area of $(\delta x = D_x/N) \times (\delta y = D_y/N')$, and the pixels are labeled by their x and y coordinates: $x_j = j\delta x$ and $y_{j'} = j'\delta y$, where $j = 0, \pm 1, \pm 2, \ldots, \pm M$ $(N = 2M + 1)$, and likewise for j'.

Since discreet pixels are being used, it is convenient to express the normalization condition of Eq. (2) in terms of a discreet sum as

$$\delta x\,\delta y \sum_{j,j'=-M}^{M} u_n^*(x_j, y_{j'}) u_{n'}(x_j, y_{j'}) \cong \delta_{n,n'}\,. \qquad (3)$$

Furthermore, I note one other property of the mode functions that will be of use later on. If one of the mode functions [say, $u_{n'}(x,y)$] is real, then not only is Eq. (3) true,

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but also

$$\delta x \delta y \sum_{j,j'=-M}^{M} u_n(x_j, y_{j'}) u_{n'}(x_j, y_{j'}) \cong \delta_{n,n'} \quad (u_{n'} \text{ real}). \tag{4}$$

Consider the arrangement shown in Fig. 1, where the signal field E_A is incident on a 50/50 beam splitter. Entering the other input port of the beam splitter is field E_B which, in the detector plane, may be written as

$$\hat{\mathrm{E}}_B^{(+)}(x,y) = \left(\frac{2\pi\hbar\varpi}{L}\right)^{1/2} \sum_l \hat{b}_l v_l(x,y). \tag{5}$$

In Eq. (5) $\hat{b}_l$ is the photon annihilation operator for the mode having spatial mode function $v_l(x,y)$. The mode functions $v_l(x,y)$ are orthogonal and satisfy the same normalization conditions as the functions $u_n(x,y)$ given in Eq. (2) but need not take the same functional form as the u_n's.

The fields leaving the beam splitter are denoted E_μ and E_ν, and for a particular choice of the beam splitter phase are given by

$$\hat{\mathrm{E}}_\mu^{(+)} = \frac{1}{\sqrt{2}} (\hat{\mathrm{E}}_A^{(+)} + \hat{\mathrm{E}}_B^{(+)}), \tag{6a}$$

$$\hat{\mathrm{E}}_\nu^{(+)} = \frac{1}{\sqrt{2}} (\hat{\mathrm{E}}_A^{(+)} - \hat{\mathrm{E}}_B^{(+)}). \tag{6b}$$

These emerging fields are incident on array detectors located equal distances behind the beam splitter.

The number of photons incident on pixel j, j' of array μ in a time T is given by the operator [17]

$$\hat{\mathrm{N}}_{\mu jj'} = \frac{cT}{2\pi\hbar\varpi} \int_{x_j-\delta x/2}^{x_j+\delta x/2} \int_{y_{j'}-\delta y/2}^{y_{j'}+\delta y/2} \times dx\, dy \hat{\mathrm{E}}_\mu^{(-)}(x,y) \hat{\mathrm{E}}_\mu^{(+)}(x,y). \tag{7}$$

The corresponding operator for array ν is of the same form, with μ replaced by ν. To evaluate this expression, combine Eqs. (1), (5), and (6a) and substitute them into Eq. (7). Making the simplifying assumption that the pixel dimensions $\delta x \times \delta y$ are small enough so that the mode functions which make up E_A and E_B are approximately constant across a given pixel, it can be shown that

$$\hat{\mathrm{N}}_{\mu jj'} = \frac{\delta x \delta y cT}{2L} \Big\{ \sum_{n,n'} \hat{a}_n^\dagger \hat{a}_{n'} u_n^*(x_j, y_{j'}) u_{n'}(x_j, y_{j'}) + \sum_{l,l'} \hat{b}_l^\dagger \hat{b}_{l'} v_l^*(x_j, y_{j'}) v_{l'}(x_j, y_{j'}) + \sum_{n,l} [\hat{a}_n^\dagger \hat{b}_l u_n^*(x_j, y_{j'}) v_l(x_j, y_{j'}) + \text{H.c.}] \Big\}, \tag{8}$$

where H.c. denotes the Hermitian conjugate. The expression for $\hat{N}_{\nu jj'}$ is nearly identical, the only difference being that the last sum (over n and l) is subtracted rather than added.

The operator of primary interest in balanced detection is that corresponding to the difference number of photons for each pixel, $\Delta\hat{N}_{jj'} = \hat{N}_{\mu jj'} - \hat{N}_{\nu jj'}$. If we let $L = cT$, this difference number is

$$\Delta\hat{N}_{jj'} = \delta x \delta y \sum_{n,l} [\hat{a}_n^\dagger \hat{b}_l u_n^*(x_j, y_{j'}) v_l(x_j, y_{j'}) + \text{H.c.}]. \tag{9}$$

Equation (9) is a general expression, which holds regardless of the states of the fields E_A and E_B. I now specialize on the case where the LO field E_B is a single-mode, plane-wave coherent state. I assume that the LO field is incident perpendicular to the detector arrays, with all of the other plane-wave modes being in the vacuum. The field state for E_B is thus $|0, \ldots, 0, \beta e^{i\phi}, 0, \ldots, 0\rangle$, where β is the amplitude of the coherent state, and ϕ is its phase. The properly normalized spatial function for this mode is

$$v_{lo}(x,y) = \frac{1}{(D_x D_y)^{1/2}}. \tag{10}$$

If the amplitude of this state is large ($\beta \gg NN'$; i.e., β is much larger than the total number of pixels) the dominant terms in Eq. (9) will be those proportional to β. Thus, it is reasonable to trace Eq. (9) over the state of the LO field, which replaces the operators for the LO field by their corresponding coherent state amplitudes

$$\Delta\hat{\mathrm{N}}_{jj'\phi} \cong \frac{\delta x \delta y}{(D_x D_y)^{1/2}} \beta \sum_n [\hat{a}_n u_n(x_j, y_{j'}) e^{-i\phi} + \hat{a}_n^\dagger u_n^*(x_j, y_{j'}) e^{i\phi}], \tag{11}$$

where the subscript ϕ indicates that this operator depends on the phase of the local oscillator. This same approximation is often made in the theory of balanced homodyne detection using nonarray detectors, and it is expected to be equally valid here (for a further discussion of this approximation, see Ref. [18]).

Now assume that one wishes to perform a measurement on a particular mode $n = m$ of the signal field; furthermore, assume that $u_m(x,y)$ is real. Multiplying Eq. (11) by $u_m(x_j, y_{j'})$ and summing over j and j', with the aid of Eqs. (3) and (4), demonstrates that

$$\sum_{j,j'=-M}^{M} \Delta\hat{\mathrm{N}}_{jj'\phi} u_m(x_j, y_{j'}) = \frac{\beta}{(D_x D_y)^{1/2}} \times [\hat{a}_m e^{-i\phi} + \hat{a}_m^\dagger e^{i\phi}]. \tag{12}$$

The term in brackets on the right side of this equation is proportional to the operator for the rotated quadrature amplitude of mode m:

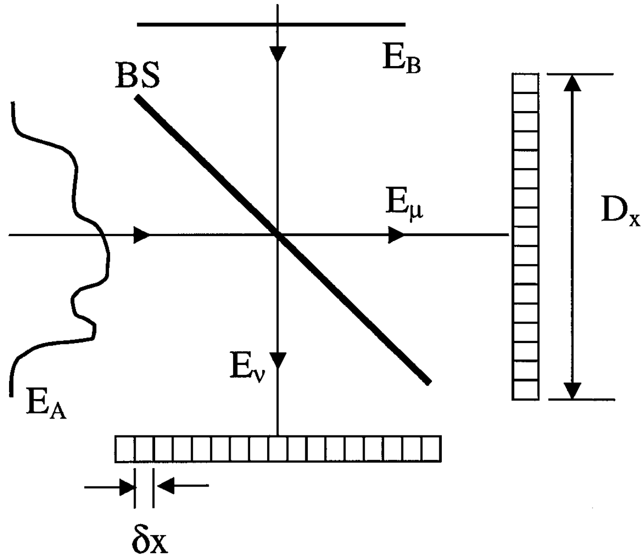

FIG. 1. The experimental apparatus; BS stands for 50/50 beam splitter. The field being measured E_A has a complicated spatial structure, while the LO field E_B is a plane wave. For clarity only the detector pixels in the x direction have been shown. The y direction is perpendicular to the page, so there are also rows of pixels in each detector above and below the plane of the page.

$$\hat{x}_{m\,\phi} = \frac{1}{\sqrt{2}}[\hat{a}_m e^{-i\phi} + \hat{a}_m^\dagger e^{i\phi}] = \frac{1}{\beta}\left(\frac{D_x D_y}{2}\right)^{1/2} \sum_{j,j'=-M}^{M} \Delta\hat{\text{N}}_{jj'\,\phi} u_m(x_j, y_{j'}) . \quad (13)$$

Equation (13) is the main result of this Letter. By simultaneously measuring the photon difference number for each pixel $\Delta N_{jj'\,\phi}$ it is possible to determine the rotated quadrature amplitude $x_{m\,\phi}$ corresponding to a particular spatial mode by combining the difference numbers according to Eq. (13). The rotation angle ϕ is varied by adjusting the phase of the LO beam. The measured mode is chosen by the selection of the spatial mode function $u_m(x, y)$, with the constraint that this mode function must be real.

It is well known that if one can perform measurements corresponding to $x_{m\,\phi}$ for $0 \leq \phi \leq \pi$, then it is possible to determine the quantum mechanical state of the field corresponding to mode m [1–5]. Thus, Eq. (13) demonstrates that an array detector is capable of making measurements which will allow one to determine the quantum mechanical state of an arbitrary mode of an optical field.

The fact that an array detector can measure the state of an optical field is not surprising. What probably is surprising is that in this detection scheme the mode functions of the measured mode $u_m(x, y)$ and of the LO mode $v_{lo}(x, y)$ are *not* the same, but this mode mismatch does not reduce the effective detection efficiency of the measurements. Thus, if the measured mode function is the same as the mode function of the actual signal ($m = s$), then the effective mode matching efficiency is unity.

The fact that there is no loss of efficiency due to LO mode mismatch when using an array detector is best illustrated by an example. Assume that the signal beam occupies the spatial mode

$$u_s(x, y) = \left(\frac{2}{D_x D_y}\right)^{1/2} \cos(kx) , \quad (14)$$

where $k = 2\pi/D_x$. The LO beam is a plane wave normal to the detector face, and its mode function is given in Eq. (10). If we perform homodyne detection of the signal mode given by Eq. (14) with this local oscillator using standard (i.e., nonarray) detectors of dimensions $D_x \times D_y$, the mode-matching efficiency η_{mm} is given by (see, for example, Ref. [4])

$$(\eta_{mm})^{1/2} = \int_{-D_x/2}^{D_x/2} \int_{-D_y/2}^{D_y/2} dx\, dy\, v_{lo}(x, y) u_s(x, y) = 0 . \quad (15)$$

So, for this particular choice of modes the signal and LO are orthogonal. A standard homodyne detector is completely insensitive to this signal field and will yield no information about it.

To demonstrate how the array detector responds to the mode given by Eq. (14), assume that this mode is in a coherent state with an amplitude of α_s, and all other modes of the signal field are in the vacuum: $|0, \ldots, 0, \alpha_s, 0, \ldots 0\rangle$. For simplicity I examine only the mean value of the detected quadrature amplitude to show that the array detector is sensitive to a field in this mode. Setting $m = s$ in Eq. (13), it is readily seen that

$$\langle \hat{x}_{s\,\phi} \rangle = \frac{1}{\beta} \sum_{j,j'=-M}^{M} \langle \Delta\hat{\text{N}}_{jj'\,\phi} \rangle \cos(kx_j) . \quad (16)$$

Using Eq. (11) it is found that

$$\langle \Delta\hat{\text{N}}_{jj'\,\phi} \rangle \cong \frac{\delta x \delta y \sqrt{2}}{D_x D_y} \beta[\alpha_s e^{-i\phi} + \alpha_s^* e^{i\phi}] \cos(kx_j) . \quad (17)$$

Substituting this expression into Eq. (16) and summing yields

$$\langle \hat{x}_{s\phi} \rangle \cong \frac{1}{\sqrt{2}} [\alpha_s e^{-i\phi} + \alpha_s^* e^{i\phi}] . \quad (18)$$

This is the same expression one would find for a standard homodyne detector with a perfectly mode-matched LO—the amplitude has not been decreased by a factor proportional to the overlap of the signal and LO spatial modes.

The detector itself yields measurements of $\Delta N_{jj'\,\phi}$, while according to Eq. (13) the quadrature amplitude $x_{m\,\phi}$ corresponding to the measured mode is determined by summing the measured values of $\Delta N_{jj'\,\phi}$ with a weighting factor given by the mode function $u_m(x, y)$. Thus, by choosing different mode functions, it is possible to determine the quadrature amplitudes of many different spatial modes for any given set of measurements $\Delta N_{jj'\phi}$. One could thus imagine taking a set of data and searching for the spatial mode which contains some desired property—the mode that has the most squeezing, for example.

Despite the fact that the quadrature amplitudes of many modes may be measured simultaneously, it is not possible to use this technique directly to measure the joint quantum state of these modes. This is because all of the modes are measured with the same rotation angle ϕ; to determine the joint quantum state each mode must have its own independently adjustable phase angle [6,7]. It may be possible, however, to modify the implementation described here to allow one to determine the full joint quantum state of two or more modes, as has been done for standard homodyne detectors [7–9].

The effect of having less than unity quantum efficiency for the detectors in the array has not been explicitly considered here, but this effect is essentially the same as for the case of nonarray detectors. If each pixel has quantum efficiency η, then the measured quantum distribution function is not simply the Wigner function but is instead an s-parametrized distribution, with $s = (1 - 1/\eta)$ [4,5].

In conclusion, I have presented an analysis of quantum state tomography based on array detectors. I have shown that it is possible to measure the rotated quadrature amplitude of a desired spatial mode using this technique; hence it is possible to determine the quantum state of this mode. The technique uses a plane-wave local oscillator beam that is not mode matched to the signal mode, but this mode mismatch does not necessarily lead to any loss of efficiency in the measurements. The measured mode is determined by the experimenter during the data analysis, and the only limitation on this mode is that its mode function must be real.

This measurement technique could prove to be extremely valuable for studying fields generated by nonlinear optical processes which produce fields having complicated spatial structure [13,14,19]. It could also be useful for studying quantum effects in optical imaging [20,21].

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[1] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).
[2] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993).
[3] D. T. Smithey, M. Beck, J. Cooper, M. G. Raymer, and A. Faridani, Phys. Scr. **T48**, 1514 (1993).
[4] U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, England, 1997).
[5] G. M. D'Ariano, in *Quantum Optics and the Spectroscopy of Solids*, edited by T. Hakioglu and A. S. Shumovsky (Kluwer, Dordrecht, 1997), p. 139.
[6] H. Kuhn, D.-G. Welsch, and W. Vogel, Phys. Rev. A **51**, 4240 (1995).
[7] M. G. Raymer, D. F. McAlister, and U. Leonhardt, Phys. Rev. A **54**, 2397 (1996).
[8] G. M. D'Ariano, M. F. Sacchi, and P. Kumar, Phys. Rev. A **61**, 013806 (2000).
[9] M. G. Raymer and A. C. Funk, Phys. Rev. A **61**, 015801 (2000).
[10] T. J. Dunn, I. A. Walmsley, and S. Mukamel, Phys. Rev. Lett. **74**, 884 (1995).
[11] D. Leibfried, D. M. Meekhof, B. E. King, C. Monroe, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **77**, 4281 (1996).
[12] S. Schiller, G. Breitenbach, S. F. Pereirs, T. Muller, and J. Mlynek, Phys. Rev. Lett. **77**, 2933 (1996).
[13] A. LaPorta and R. E. Slusher, Phys. Rev. A **44**, 2013 (1991).
[14] S. K. Chio, R. D. Li, C. Kim, and P. Kumar, J. Opt. Soc. Am. B **14**, 1564 (1997).
[15] C. Kim and P. Kumar, Phys. Rev. Lett. **73**, 1605 (1994).
[16] E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970), p. 149.
[17] M. G. Raymer, J. Cooper, and M. Beck, Phys. Rev. A **48**, 4617 (1993).
[18] S. L. Braunstein, Phys. Rev. A **42**, 474 (1990).
[19] A. Beržanskis, W. Chinaglia, L. A. Lugiato, K.-H. Feller, and P. Di Trapani, Phys. Rev. A **60**, 1622 (1999).
[20] M. Marable, S.-K. Choi, and P. Kumar, Opt. Express **2**, 84 (1998).
[21] A. Gatti, E. Brambilla, L. A. Lugiato, and M. I. Kolobov, Phys. Rev. Lett. **83**, 1763 (1999).