

## Bose-Einstein Condensates with $1/r$ Interatomic Attraction: Electromagnetically Induced “Gravity”

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We show that particular configurations of intense off-resonant laser beams can give rise to an attractive  $1/r$  interatomic potential between atoms located well within the laser wavelength. Such a “gravitational-like” interaction is shown to give stable Bose-Einstein condensates that are *self-bound* (without an additional trap) with unique scaling properties and measurably distinct signatures.

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In the atomic Bose-Einstein condensates (BECs) created thus far [1] the atoms interact only at very short distance in good correspondence with the hard-sphere model. The majority of the properties of these dilute gases can be understood by taking into account only two-body collisions which are characterized by the  $s$ -wave scattering length [2]. A number of groups [3] have investigated the fascinating possibility of changing the magnitude and sign of the  $s$ -wave scattering length using external fields. The resulting condensates retain the essential hard-sphere,  $s$ -wave, nature of the interatomic interaction.

Here we introduce a qualitatively new regime of cold atoms in which the atom-atom interactions are attractive and have a very long range, varying as  $r^{-1}$  [4].

We shall demonstrate that a stable BEC with attractive  $r^{-1}$  interactions is achievable by irradiating the atoms with intense, *extremely off-resonant*, electromagnetic fields. The atoms are then coupled via the dipoles that are induced by these external fields (in contrast to those induced by the random vacuum field responsible for the van der Waals–London interaction, which varies as  $r^{-6}$ , and leads to the usual hard-sphere description) [5].

Such an  $r^{-1}$  attractive potential can simulate gravity between quantum particles. Remarkably, this potential gives an interatomic attraction (depending on the laser intensity and wavelength) which can be some 17 orders of magnitude greater than their gravitational interaction at the same distance.

This suggests it might be possible to study gravitational effects, normally only important on the stellar scale, in the laboratory. Particularly interesting is the possibility of experimentally emulating boson stars [6]: gravitationally bound condensed boson configurations of finite volume, in which the zero-point kinetic energy balances the gravitational attraction and thus *stabilizes* the system against collapse.

In this Letter we shall discuss the interplay of the usual hard-core interatomic potential with an electromagnetically induced “gravitational” one on a BEC using a variational mean-field approximation (MFA). Two new physical

regimes with unique scaling properties emerge where the BEC is self-bound (no trap required).

How can one realize the  $r^{-1}$  potential between neutral atoms? Consider the dipole-dipole interaction energy induced by the presence of external electromagnetic radiation of intensity  $I$ , wave vector  $\mathbf{q}$ , and polarization  $\hat{\mathbf{e}}$ . This energy can be written (within fourth order perturbation theory) in terms of Cartesian components  $i, j$  [5] (in S.I. units)

$$U(\mathbf{r}) = \left( \frac{I}{4\pi c \epsilon_0^2} \right) \alpha^2(q) \hat{\mathbf{e}}_i^* \hat{\mathbf{e}}_j V_{ij}(q, \mathbf{r}) \cos(\mathbf{q} \cdot \mathbf{r}). \quad (1)$$

Here  $\mathbf{r}$  is the interatomic axis,  $\alpha(q)$  the isotropic, dynamic, polarizability of the atoms at frequency  $cq$ , and  $V_{ij}$  is the retarded dipole-dipole interaction tensor

$$V_{ij} = \frac{1}{r^3} [(\delta_{ij} - 3\hat{\mathbf{r}}_i \hat{\mathbf{r}}_j) (\cos qr + qr \sin qr) - (\delta_{ij} - \hat{\mathbf{r}}_i \hat{\mathbf{r}}_j) q^2 r^2 \cos qr], \quad (2)$$

where  $\hat{\mathbf{r}}_i = r_i/r$ . We stress that the atoms exchange only virtual photons and remain unexcited. For a *fixed orientation* of the interatomic axis with respect to the external field, Eqs. (1) and (2) give the well known  $r^{-3}$  variation of the interaction energy at near-zone separations ( $qr \ll 1$ ). The near-zone limit of  $U(\mathbf{r})$  is *strongly anisotropic*. It was noted by Thirunamachandran [5] that when an *average over all orientations* of the interatomic axis with respect to the incident radiation direction is taken, the static dipolar part of the coupling [i.e., the instantaneous, non-retarded part  $r^{-3}(\delta_{ij} - 3\hat{\mathbf{r}}_i \hat{\mathbf{r}}_j)$ ] vanishes. The remaining “transverse” part is, in the near zone, an attractive  $r^{-1}$  potential. It is weaker by a factor of  $(qr)^2$  than the  $r^{-3}$  term.

However, thus far no scheme has been suggested wherein an average over all orientations is guaranteed for cold gases. We shall consider a spatial configuration of external fields which *enforces the “averaging out”* of the  $r^{-3}$  interactions. A simple combination, which ensures the suppression of the  $r^{-3}$  interaction while retaining the weaker  $r^{-1}$  attraction in the near zone, uses three orthogonal circularly polarized laser beams pointing along  $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$  (“a triad”—see Fig. 1). Let us momentarily ignore

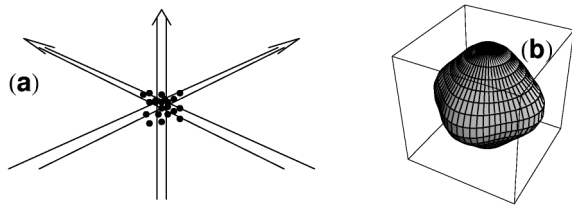


FIG. 1. (a) Schematic depiction of a triad of lasers incident upon an ensemble of atoms. This triad generates the attractive  $r^{-1}$  potential given by Eq. (3), whose magnitude has the angular dependence shown in (b).

interference between the three beams, and consider only the sum of their intensities. In the near zone one can Taylor expand Eqs. (1) and (2) in powers of the small quantity  $qr$ . Using the identity  $\hat{\mathbf{e}}_i^{*(\pm)}(\mathbf{q})\hat{\mathbf{e}}_j^{(\pm)}(\mathbf{q}) = \frac{1}{2}[(\delta_{ij} - \hat{\mathbf{q}}_i\hat{\mathbf{q}}_j) \pm i\epsilon_{ijk}\hat{\mathbf{q}}_k]$ , with  $+$  ( $-$ ) corresponding to left (right) circular polarizations, together with Eqs. (1) and (2), the triad can be shown to give rise to the (near-zone)  $r^{-1}$  pair potential

$$U(\mathbf{r}) = -\frac{3Iq^2\alpha^2}{(16\pi c\epsilon_0^2)} \times \frac{1}{r} \left[ \frac{7}{3} + (\sin\theta \cos\phi)^4 + (\sin\theta \sin\phi)^4 + (\cos\theta)^4 \right]. \quad (3)$$

Note that this interaction is attractive for any orientation  $(\theta, \phi)$  of  $\mathbf{r}$  relative to the beams as long as the polarizability  $\alpha(q)$  is real.

If one wishes, the angular anisotropy in (3) can be canceled to give a purely radial  $r^{-1}$  potential by combining a number of such triads with different orientations. It is convenient to define the orientation of each triad by the Euler angles  $(\alpha, \beta, \gamma)$  [7], namely, a rotation of  $\alpha$  about the  $\hat{\mathbf{z}}$  axis, followed by a rotation of  $\beta$  about the new  $\hat{\mathbf{y}}$  axis and finally a rotation of  $\gamma$  about the final  $\hat{\mathbf{z}}$  axis. One configuration that cancels the anisotropy completely uses six triads (18 laser beams) rotated through the following Euler angles:  $(0, \pi/4, \pi/8)$ ,  $(0, \pi/4, -\pi/8)$ ,  $(0, \pi/4, 3\pi/8)$ ,  $(0, \pi/4, -3\pi/8)$ ,  $(0, 0, \pi/8)$ ,  $(0, 0, -\pi/8)$ . The last two triads should be of half the intensity  $I$  of the others. Then the interatomic potential becomes

$$U(\mathbf{r}) = -\frac{11}{4\pi} \frac{Iq^2\alpha^2}{c\epsilon_0^2} \frac{1}{r} = -\frac{u}{r}. \quad (4)$$

The main difficulty in realizing the near-zone  $r^{-1}$  potential is that the  $r^{-3}$  interaction survives due to the *interference* between different pairs of beams, whose contribution is proportional to the product of their respective field amplitudes. This difficulty can be overcome if one introduces frequency shifts between the laser beams. Spreading the frequencies  $\omega_n$  of the  $E_n$  laser fields ( $n = 1, 2, 3$  for one triad or  $n = 1, 2, \dots, 18$  for six triads) in intervals about the central frequency makes the  $r^{-3}$  interference terms in the interaction energy  $\propto E_n E_{n'}^*$  ( $n \neq n'$ ) oscillate

at the difference frequencies  $|\omega_n - \omega_{n'}|$ . If these difference frequencies are much higher than the other relevant frequencies (e.g., collective oscillation frequencies), then the contribution of the interference terms to the mean-field potential averages to zero. Typically these conditions hold for  $|\omega_n - \omega_{n'}| \geq 10^4$  Hz. Angular misalignment errors,  $\delta$ , between the orthogonal beams should satisfy  $\delta \ll qL$  (where  $L$  is the mean radius of the condensate) and intensity fluctuations should satisfy  $\Delta I/I \ll qL$ , in order to ensure the  $r^{-3}$  cancellation for the noninterfering terms  $\propto \sum_n |E_n|^2$ . Although these oscillating  $r^{-3}$  terms do not contribute to the mean-field potential, they can eject atoms from the condensate, but this process can be strongly reduced, as will be discussed at the end of this Letter.

A *lower bound* on the magnitude of the  $r^{-1}$  attraction is obtained by using the static rather than dynamic polarizability, which for sodium atoms, say, has the value  $24.08 \times 10^{-24}$  cm<sup>3</sup>. Thus, for strongly off-resonant light, that from a CO<sub>2</sub> laser ( $q = 2\pi/10.6$   $\mu\text{m}$ ) say, of intensity  $I = 10^8$  W/cm<sup>2</sup>, one finds  $-u/r \approx -2 \times 10^{-15}$  eV, at  $r = 100$  nm, the mean separation in a typical BEC. This is comparable to the magnitude of the van der Waals–London dispersion energy at this distance. However, in a system of many atoms the  $r^{-1}$  potential acts over the entire sample, whereas the van der Waals–London interaction is effective only for nearest neighbors, and the  $r^{-1}$  contribution to the total energy can become important.

Our treatment of the many particle problem is based on a two-body potential  $V(\mathbf{r}) = 4\pi a\hbar^2\delta(\mathbf{r})/m - u/r$ , where the first term is the pseudopotential arising from the  $s$ -wave scattering ( $a$  is the  $s$ -wave scattering length and  $m$  the atomic mass). In order to write  $V$  in this form we require that the  $-u/r$  potential be sufficiently weak (compared with the mean kinetic energy per particle) so as not to affect the short-range hard-sphere scattering. This requirement certainly holds if  $a_* \gg \lambda_{\text{DB}} \gg a$ , where  $\lambda_{\text{DB}}$  is the de Broglie wavelength and  $a_* = \hbar^2/mu$  is the Bohr radius associated with the gravitational-like coupling  $u$ . With the values given above  $a_* \sim 10$  cm, while for a typical BEC  $\lambda_{\text{DB}} \sim 10^{-5} - 10^{-3}$  m and  $a \sim 3$  nm.

Consider now the application of this two-body potential to a trapped dilute BEC gas well below the critical temperature. We assume that the condensate initially occupies a fraction of the wavelength of the laser so that the near-zone condition is valid (lasers in the far infrared, or microwave sources would satisfy this condition). The “zero-temperature” many-body problem leads, within the MFA, to the following equation for the order parameter  $\Psi(\mathbf{R}, t)$ :

$$i\hbar \frac{\partial \Psi(\mathbf{R}, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{R}) + V_{\text{H}}(\mathbf{R}) \right] \Psi(\mathbf{R}, t), \quad (5)$$

where  $V_{\text{ext}}(R) = m\omega_0^2 R^2/2$  is for simplicity an isotropic trap potential (which can be set to zero in certain cases—see below), and  $V_{\text{H}}(\mathbf{R})$  is the self-consistent

Hartree potential

$$V_H(\mathbf{R}) = \frac{4\pi a\hbar^2}{m} |\Psi(\mathbf{R}, t)|^2 - u \int d^3R' \frac{|\Psi(\mathbf{R}', t)|^2}{|\mathbf{R}' - \mathbf{R}|}. \quad (6)$$

The order parameter  $\Psi(\mathbf{R}, t)$  is normalized so that  $\int d^3R |\Psi(\mathbf{R}, t)|^2 = N$ , with  $N$  the total number of atoms. The usual Gross-Pitaevskii (GP) equation [2] is recovered in the limit when  $u = 0$ . The MFA is valid when the system is dilute, i.e.,  $\rho a^3 \ll 1$ , with  $\rho$  the density. An additional condition on the MFA validity is that the  $r^{-1}$  potential is weak, and this can be expressed as  $\rho a_*^3 \gg 1$ . This constraint, as in the related problem of the charged boson gas [8], means that many atoms must be present within the interaction volume  $a_*^3$ . However, it is, in fact, the diluteness condition that turns out to be the stricter of the two, as one can check from the MFA results for the density (see Table I below).

An analytical estimate for the mean radius of the  $N$ -atom condensate can be given using the following variational wave function

$$\Psi_\lambda(R) = N^{1/2} (\pi \lambda^2 l_0^2)^{-3/4} \exp(-R^2/2\lambda^2 l_0^2), \quad (7)$$

where  $l_0 = \sqrt{\hbar/m\omega_0}$ . The variational parameter  $\lambda$  is proportional to the root mean square radius through  $\sqrt{\langle R^2 \rangle} = \sqrt{3/2} \lambda l_0$ . This parameter is obtained by minimizing the variational mean-field energy

$$\frac{H(\lambda)}{N \hbar \omega_0} = \frac{3}{4} \left( \lambda^{-2} + \lambda^2 - 2\tilde{u}\lambda^{-1} + \frac{2}{3} \tilde{s}\lambda^{-3} \right), \quad (8)$$

where we have chosen the dimensionless  $\tilde{u}$  (proportional to the ‘‘gravity’’ strength  $u$ ) and  $\tilde{s}$  (proportional to  $s$ -wave scattering length  $a$ ) to be

$$\begin{aligned} \tilde{u} &= \pi \sqrt{32\pi/9} (N l_0 / a_*), \\ \tilde{s} &= \sqrt{2/\pi} (N a / l_0). \end{aligned} \quad (9)$$

The numerical factors are chosen to make the equation for  $\lambda$  simple

$$-\lambda^{-4} + 1 + \tilde{u}\lambda^{-3} - \tilde{s}\lambda^{-5} = 0. \quad (10)$$

This equation is equivalent to requiring that the variational solution satisfies the following virial relation:  $-T +$

TABLE I. A comparison of the four asymptotic regions. The radius,  $\lambda$ , and the release energy,  $E_{\text{rel}}$ , are discussed in the text.  $\rho_{\text{max}}$  is the peak density (at the center) of the condensate.

	$G$	TF- $G$	TF- $O$	$I$
defn.:	$\tilde{u} \gg 1$ $\tilde{s} \ll 1/\tilde{u}$	$\tilde{s} \ll \tilde{u}^{5/3}$ $\tilde{s} \gg 1/\tilde{u}$	$\tilde{s} \gg 1$ $\tilde{s} \gg \tilde{u}^{5/3}$	$\tilde{u} \ll 1$ $\tilde{s} \ll 1$
$\lambda$ :	$1/\tilde{u}$	$(\tilde{s}/\tilde{u})^{1/2}$	$\tilde{s}^{1/5}$	1
$E_{\text{rel}}/\hbar\omega_0$ :	$\frac{3}{4}\tilde{u}^2 N$ $\propto N^3$	$\frac{1}{2}\tilde{u}^{3/2}\tilde{s}^{-1/2} N$ $\propto N^2$	$\frac{1}{2}\tilde{s}^{2/5} N$ $\propto N^{7/5}$	$\frac{3}{4} N$ $\propto N$
$\rho_{\text{max}}$ :	$\frac{\sqrt{32^3}\pi^3 N^4}{27a_*^3}$	$\frac{2\pi^2 N}{\sqrt{aa_*^3}}$	$\frac{15^{2/5} N^{2/5}}{8\pi a_*^{3/5} l_0^{12/5}}$	$\frac{N}{\sqrt{\pi^3} l_0^3}$

$V_{\text{ext}} - \frac{1}{2}U_u - \frac{3}{2}U_s = 0$ , where  $T$ ,  $V_{\text{ext}}$ ,  $U_u$ , and  $U_s$  are the kinetic energy, the harmonic trap potential energy, and the internal energies due to the  $-u/r$  and hard-sphere interatomic potentials, respectively. This relation can be obtained from scaling considerations (see Ref. [9] for the case  $u = 0$ ).

The general asymptotic properties of the ground state solutions of Eqs. (5) and (6), as a function of  $(\tilde{u}, \tilde{s})$ , are summarized in the ‘‘phase diagram’’ of Fig. 2a for positive scattering lengths. In this diagram there are four asymptotic regions: The noninteracting ideal region ( $I$ ) and the ordinary Thomas-Fermi region (TF- $O$ ) are dominated by the balance of the external trap potential with, respectively, the kinetic energy and the repulsive  $s$ -wave scattering, and so are not sensitive to the  $-u/r$  potential. The regions  $G$  and TF- $G$ , which represent two *new physical regimes* for atomic BECs, are controlled by the balance of the gravity-like potential with either the kinetic energy ( $G$ ) or the  $s$ -wave scattering (TF- $G$ ). Neither region is sensitive to the external trap, so that we can adiabatically turn it off ( $V_{\text{ext}} = 0$ ) and access either the  $G$  or the TF- $G$  region. Experimentally, direct signatures of the  $r^{-1}$  interaction come from the radius  $\lambda$  and the release energy  $E_{\text{rel}} = T + U_s$ . The release energy is the kinetic energy that can be measured after the expansion occurring due to switching off the external trap and the laser fields [2]. Table I summarizes these quantities as well as the peak density  $\rho_{\text{max}}$  in the four regions. We now focus on the properties of the two new regimes: (a) In the TF- $G$  region an analytic solution for the ground state of Eqs. (5) and (6) is given by

$$\Psi_{\text{TF-G}}(\mathbf{R}) = \frac{\sqrt{N}}{2R_0} \sqrt{\frac{\sin(\pi R/R_0)}{R}} \Theta(R_0 - R), \quad (11)$$

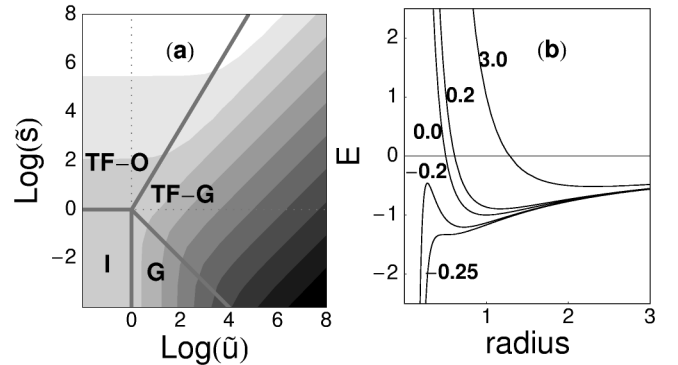


FIG. 2. (a): Contour plot of  $\log(\lambda)$ , where  $\lambda$  is the condensate radius, in the parameter space  $\log(\tilde{u})$  versus  $\log(\tilde{s})$ : darker shade corresponds to smaller  $\lambda$ . The border separating the TF- $O$  and TF- $G$  regions is given by  $\tilde{s} = \tilde{u}^{5/3}$  and that separating the TF- $G$  and  $G$  regions by  $\tilde{s} = \tilde{u}^{-1}$ . (b) Mean energies per particle for large  $\tilde{u}$  (no external trap) as a function of the condensate radius. Curves are plotted for positive as well as negative values of the scattering strength  $\tilde{s}\tilde{u}$ . For  $\tilde{s}\tilde{u} \leq -1/4$  there is no minimum for a finite radius. The energy and radii units are  $\tilde{u}^2 \hbar \omega_0$  and  $l_0/\tilde{u}$ , respectively.

where  $R_0 = \sqrt{aa^*}/2$ . Contrary to the ordinary Thomas-Fermi limit of the GP equation, the size of the condensate is fixed by the ratio of the coupling constants,  $4\pi a\hbar^2/m$  and  $u$ , and is *independent of  $N$* . (b) The  $G$  region, where only the  $r^{-1}$  attraction and kinetic energy play a role, is of particular interest since our system is then equivalent to a boson star (a system of gravitating bosons) [6] in the nonrelativistic regime. The mean-field equations in this region are also identical to those describing a single particle moving in the gravitational field generated by its own wave function [10]. In both cases smooth bound solutions have been shown to exist [6,10]. This establishes the possibility of a stable *self-bound* (no external trap)  $r^{-1}$  condensate.

The gravitational-like attraction does not induce “collapse” of the condensate, since, at short radii, it is always weaker than the kinetic energy. This can be seen from the scaling of the kinetic energy ( $\lambda^{-2}$ ) versus that of the  $r^{-1}$  potential ( $\lambda^{-1}$ ) in Eq. (8) and Fig. 2b. By contrast, this kind of instability can occur for *negative* scattering lengths [3,4] when  $N$  exceeds a critical number ( $N_{cr} \approx 0.6 \times l_0/|a|$ ) because the mean energy due to scattering ( $\lambda^{-3}$ ) is dominant at small radii. The  $u/r$  attraction does reduce, when combined with the attractive scattering, the critical number to  $N_{cr} \approx 0.17 \times \sqrt{a^*/|a|}$  (see Fig. 2b for the critical case with  $\tilde{s}\tilde{u} = -1/4$ ).

Now we estimate the losses of  $G$  or TF- $G$  condensates due to the  $r^{-3}$  oscillating *interfering* terms discussed above. Consider one of the possible oscillating interfering terms  $A(\mathbf{r})\cos(\Omega t)$ , where  $A(x, y, z) = -3u\frac{xy}{q^2 r^3}$  and  $\Omega$  is the difference in frequency between the two interfering lasers. Fermi’s golden rule gives an expression for the rate of depletion of the condensate density  $|\Psi|^2$  due to creation of a pair of quasiparticles of opposite momenta (with  $k \approx \pm\sqrt{m\Omega/\hbar}$ ) in an homogeneous Bose gas:  $d|\Psi|^2/dt = -(|A(k)|^2/6\pi)|\Psi|^4(m/\hbar^2)^{3/2}\sqrt{\Omega/\hbar}$ , where  $|A(k)|^2$  is the angular average of the square of the Fourier transform of  $A(\mathbf{r})$  ( $0.1418 u^2/q^4$ ). Applied to our problem, we find the following approximations:  $dN_0/dt \approx \tilde{u}^5\sqrt{\Omega\omega_0}/(ql_0)^4$  in the  $G$  region and  $dN_0/dt \approx \tilde{u}^{7/2}\tilde{s}^{-3/2}\sqrt{\Omega\omega_0}/(ql_0)^4$  in the TF- $G$  region. These expressions can be used to find conditions such that these loss rates are smaller than, say, the trap oscillation frequency  $\omega_0$ . Taking, e.g.,  $\Omega \approx 2\pi \times 10^4 \text{ s}^{-1}$ ,  $\omega_0 \approx 2\pi \times 10^2 \text{ s}^{-1}$ ,  $ql_0 \approx 1$ ,  $\tilde{u} \approx 5$ , we obtain the following: For the  $G$  region  $dN_0/dt \approx 6 \times 10^4 \omega_0$ ; i.e., we need more than  $10^5$  atoms. For the TF- $G$  region (e.g.,  $\tilde{s} \approx 1$ ),  $dN_0/dt \approx 6 \times 10^3 \omega_0$ ; i.e., we need more than  $10^4$  atoms.

Finally, we note that the collective Rayleigh scattering of the off-resonant laser photons can in part excite phonons in the condensate [11]. In our estimation the rate of depletion due to Rayleigh scattering becomes sufficiently small

(much smaller than the mean-field energy per particle divided by  $\hbar$ ) when the system radius is smaller than the laser wavelength ( $q\sqrt{\langle R^2 \rangle} \ll 1$ ).

To conclude, the laser-induced attractive  $r^{-1}$  interaction can give rise to stable condensates with unique static properties. Their stability, long lifetime (low loss rates incurred by the  $r^{-3}$  oscillating terms) and lack of sensitivity to alignment errors or amplitude noise of the laser beams makes the experimental realization of such condensates rather likely. Their fascinating analogy with gravitating quantum systems whose gravitational interaction can be enormously enhanced by the field merits further research.

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