Quasilinear Theory of the 2D Euler Equation

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We develop a quasilinear theory of the 2D Euler equation and derive an integrodifferential equation for the evolution of the coarse-grained vorticity $\overline{\omega}(\mathbf{r}, t)$. This equation respects all of the invariance properties of the Euler equation and conserves angular momentum in a circular domain and linear impulse in a channel. We show under which hypothesis we can derive an H theorem for the Fermi-Dirac entropy and make the connection with statistical theories of 2D turbulence.

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Two-dimensional (2D) flows with high Reynolds numbers have the striking property of organizing spontaneously into large-scale coherent vortices [1]. The robustness of Jupiter's great red spot, a huge vortex persisting for more than three centuries in a turbulent shear between two zonal jets, is probably related to this general phenomenon. Many other vortices are observed in geophysical and astrophysical flows. Understanding the structure and formation of these organized states is still a challenging problem.

To be explicit, we consider as an initial condition a stripe of uniform vorticity $\omega = \sigma_0$ surrounded by irrotational flow $\omega = 0$. This stripe is unstable and generates a complicated mixing process leading to the formation of a quasistationary vortex slightly diffusing with viscosity. This is the classical shear layer (or Kelvin-Helmholtz) instability investigated numerically in, e.g., Ref. [2]. These authors proposed to interpret the quasiequilibrium state as a state of maximum entropy under the constraint of a fixed energy and circulation. This is motivated by the statistical theory of the 2D Euler equation developed by Miller [3] and Robert and Sommeria [4]. A coarse-graining procedure is introduced and a mixing entropy is constructed to describe the chaotic interchange of vorticity levels along the evolution. Since the vorticity levels cannot overlap, they follow an exclusion principle and this leads to the statistics of a Fermi-Dirac-type. Comparision with numerical simulations [2] shows very good agreement with the theoretical prediction in the core of the vortex, where the fluctuations are sufficient to validate the ergodicity hypothesis. This statistical mechanics of phase mixing is closely related to the theory of "violent relaxation" developed by Lynden-Bell [5] for collisionless stellar systems (e.g., elliptical galaxies) described by the Vlasov equation [6,7].

Less is known concerning the relaxation towards equilibrium. This is clearly a complicated task and analytical results will be obtained only by introducing approximations. Our objective is to derive a kinetic equation respecting all of the conservation laws and invariance properties of the Euler equation and driving the system towards the Fermi-Dirac state by increasing the mixing entropy. If such a program can be realized this will provide a useful subgrid scale model, allowing large eddy simulations (LES) of 2D turbulence with potential applications in geophysics [8]. The first step in this direction was made by Robert and Sommeria [9] using a *variational* procedure. They assumed that "out of equilibrium, the system evolves so as to maximize the rate of entropy production $\dot{S}$ while respecting all the constraints of the Euler equation." This maximum entropy production principle (MEPP) leads to an equation for the coarse-grained vorticity of a generalized Fokker-Planck-type which can be compared successfully with direct Navier-Stokes simulations [9,10]. Their method was extended by Chavanis and Sommeria [11] who derived a set of equations respecting, in addition, the invariance properties of the Euler equation. However, the MEPP is relatively *ad hoc* and *assumes* that the system evolves towards a maximum entropy state. In this Letter, we consider a completely different approach based on a *perturbative* expansion of the Euler equation. This is the counterpart of the quasilinear theory introduced in plasma physics and in stellar dynamics for the Vlasov equation [12–14]. Within some approximations, we derive a new kinetic equation of a generalized Landau-type for the coarse-grained vorticity and prove an H theorem for the Fermi-Dirac entropy (instead of postulating it). The results of the MEPP are recovered as an approximation of our model.

For a two-dimensional incompressible and inviscid flow, the Euler equation can be written as

$$\frac{\partial \omega}{\partial t} + \mathbf{u}\nabla\omega = 0, \qquad (1)$$

$$\mathbf{u} = -\mathbf{z} \wedge \nabla\psi, \qquad \omega = -\Delta\psi, \qquad (2)$$

where $\omega\mathbf{z} = \nabla \wedge \mathbf{u}$ is the vorticity and ψ is the stream function ($\mathbf{z}$ is a unit vector normal to the flow). The velocity can be expressed as an integral over the vorticity field as

$$\mathbf{u}(\mathbf{r}, t) = \int d^2\mathbf{r}'\, \mathbf{V}(\mathbf{r}' \to \mathbf{r})\omega(\mathbf{r}', t) \qquad (3)$$

where

$$\mathbf{V}(\mathbf{r}' \to \mathbf{r}) = -\frac{1}{2\pi}\frac{(\mathbf{r}' - \mathbf{r})_\perp}{|\mathbf{r}' - \mathbf{r}|^2} + \mathbf{V}_b(\mathbf{r}' \to \mathbf{r}) \qquad (4)$$

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represents the velocity created in $\mathbf{r}$ by a vortex of unit circulation located in $\mathbf{r}'$ ($\mathbf{r}_\perp$ is the vector $\mathbf{r}$ rotated by $+\frac{\pi}{2}$). The term $\mathbf{V}_b(\mathbf{r}' \to \mathbf{r})$ accounts for boundary effects ($\mathbf{V}_b = 0$ in an infinite domain) and can be calculated with the method of "images."

In the situation described previously, the Euler equation builds up an intricate filamentation at smaller and smaller scales. If we subdivide our domain into a lattice of macrocells of size ϵ, only the "coarse-grained" vorticity $\overline{\omega}(\mathbf{r}, t)$ can reach a stationary state. This coarse-grained vorticity is defined by a double averaging process [13]: a space average over the cell of size ϵ^2 centered on $\mathbf{r}^{(i)}$ and a statistical average to express our ignorance of the precise manner in which the phase filaments of vorticity σ_0 are distributed in the macrocell:

$$\overline{\omega}(\mathbf{r}^{(i)}) = \left\langle \frac{1}{\epsilon^2} \int_\epsilon \omega(\mathbf{r}^{(i)} + \mathbf{r}')\, d^2\mathbf{r}' \right\rangle. \qquad (5)$$

The fluctuating vorticity $\tilde{\omega} = \omega - \overline{\omega}$, satisfying $\overline{\tilde{\omega}} = 0$, is simply the difference between the exact vorticity and the smoothed-out vorticity. The passage from discrete to continuous variables for $\overline{\omega}$ requires a hypothesis of scale separation. We shall assume that the velocity field $\mathbf{u}(\mathbf{r}, t)$ consists of a strong large-scale component and a weak small-scale component such that the characteristic scale of $\overline{\mathbf{u}}$ is much greater than that of $\tilde{\mathbf{u}}$ (we also assume that $\tilde{\omega} \sim \overline{\omega}$). This hypothesis of scale separation was previously made by Dubrulle and Nazarenko [15]. This is of course an idealization since the energy spectrum never presents a clear-cut gap in practice. However, this approximation should account reasonably well for the nonlocal interactions between large eddies and small scale fluctuations in 2D turbulence. A more general study relaxing this hypothesis can be found in [16].

If we take the local average of the Euler equation (1), we obtain a convection-diffusion equation:

$$\frac{\partial \overline{\omega}}{\partial t} + \overline{\mathbf{u}} \nabla \overline{\omega} = -\nabla \mathbf{J} \qquad (6)$$

for the coarse-grained field with a current $\mathbf{J} = \overline{\tilde{\omega} \tilde{\mathbf{u}}}$ related to the correlations of the fine-grained fluctuations. In turn, the fluctuations depend on the smoothed-out field according to the equation

$$\frac{\partial \tilde{\omega}}{\partial t} + \overline{\mathbf{u}} \nabla \tilde{\omega} = -\tilde{\mathbf{u}} \nabla \overline{\omega} - \tilde{\mathbf{u}} \nabla \tilde{\omega} + \overline{\tilde{\mathbf{u}} \nabla \tilde{\omega}} \qquad (7)$$

obtained by subtracting (1) and (6). Within the scale separation hypothesis, we can neglect the nonlinear terms $\tilde{\mathbf{u}}\nabla\tilde{\omega}$ and $\overline{\tilde{\mathbf{u}}\nabla\tilde{\omega}}$ which represent the interactions of the small turbulent scales among themselves [15]. However, unlike [15], we keep the linear term $\tilde{\mathbf{u}}\nabla\overline{\omega}$ which takes into account the interactions between small and large scales. Its order of magnitude $\overline{\omega}^2\epsilon/L$ (where $L \gg \epsilon$ is the domain size) is relatively small but this term has a cumulative effect [see Eq. (13)], giving rise to a diffusion process. We consider therefore the coupled system,

$$\frac{\partial \overline{\omega}}{\partial t} + L\overline{\omega} = -\nabla \overline{\tilde{\omega} \tilde{\mathbf{u}}}, \qquad (8)$$

$$\frac{\partial \tilde{\omega}}{\partial t} + L\tilde{\omega} = -\tilde{\mathbf{u}} \nabla \overline{\omega}, \qquad (9)$$

where $L = \overline{\mathbf{u}}\nabla$ is an advection operator. This "quasilinear approximation" is standard in plasma physics and in stellar dynamics for the Vlasov-Poisson system [12–14] but, to our knowledge, it has never been applied to the 2D Euler system. Owing to the various approximations introduced, this theory can describe only the late quiescent stages of the relaxation when the fluctuations weaken.

By introducing the Greenian,

$$G(t_2, t_1) \equiv \exp\left\{ -\int_{t_1}^{t_2} dt L(t) \right\}, \qquad (10)$$

we can immediately write down a formal solution of (9), namely,

$$\tilde{\omega}(\mathbf{r}, t) = G(t, 0)\tilde{\omega}(\mathbf{r}, 0) - \int_0^t ds\, G(t, t - s) \times \tilde{\mathbf{u}}(\mathbf{r}, t - s)\nabla\overline{\omega}(\mathbf{r}, t - s). \qquad (11)$$

Although very compact, this formal expression is in fact extremely complicated. Indeed, all of the difficulty is encapsulated in the Greenian $G(t, t - s)$ which supposes that we can solve the smoothed out Lagrangien flow,

$$\frac{d\mathbf{r}}{dt} = \overline{\mathbf{u}}(\mathbf{r}, t), \qquad (12)$$

between t and $t - s$.

The objective now is to substitute the formal result (11) back into (8) and make a closure approximation in order to obtain a self-consistant equation for $\overline{\omega}(\mathbf{r}, t)$. If the vorticity were purely advected by the stochastic velocity field $\mathbf{u}$ (like a passive scalar), the interaction (3) would be switched off and we would end up with a diffusion equation for $\overline{\omega}$ with a diffusion coefficient $D \sim \frac{1}{4}\tau\overline{\tilde{\mathbf{u}}^2}$, where τ is the decorrelation time [6,10]. However, in the case of the Euler equation, the velocity fluctuations are induced by the fluctuations of the vorticity itself according to

$$\tilde{\mathbf{u}}(\mathbf{r}, t) = \lambda \int d^2\mathbf{r}'\, \mathbf{V}(\mathbf{r}' \to \mathbf{r})\tilde{\omega}(\mathbf{r}', t). \qquad (13)$$

Therefore, considering (11) and (13), we see that the vorticity fluctuations $\tilde{\omega}(\mathbf{r}, t)$ are given by an iterative process: $\tilde{\omega}(t)$ depends on $\tilde{\mathbf{u}}(t - s)$ which itself depends on $\tilde{\omega}(t - s)$, etc. Since $|\tilde{\mathbf{u}}|$, of order $\overline{\omega}\epsilon$, is much smaller than $|\overline{\mathbf{u}}|$, of order $L\overline{\omega}$, we can solve this problem perturbatively. This is the equivalent of the "weak coupling approximation" in plasma physics [12–14]. For convenience, we have introduced a counting parameter λ in (13) which will be set equal to 1 ultimately. To order λ^2, we obtain, after some rearrangements,

$$\frac{\partial\overline{\omega}}{\partial t} + L\overline{\omega} = \frac{\partial}{\partial r^{\mu}}\int_0^t ds \int d^2\mathbf{r}'\, d^2\mathbf{r}''\, V^{\mu}(\mathbf{r}' \to \mathbf{r})G'(t,t-s)G(t,t-s) \times \Big\{V^{\nu}(\mathbf{r}'' \to \mathbf{r})\overline{\tilde{\omega}(\mathbf{r}',t-s)\tilde{\omega}(\mathbf{r}'',t-s)}\,\frac{\partial\overline{\omega}}{\partial r^{\nu}}(\mathbf{r},t-s) + V^{\nu}(\mathbf{r}'' \to \mathbf{r}')\overline{\tilde{\omega}(\mathbf{r},t-s)\tilde{\omega}(\mathbf{r}'',t-s)}\,\frac{\partial\overline{\omega}}{\partial r'^{\nu}}(\mathbf{r}',t-s)\Big\}. \qquad (14)$$

In this expression, the Greenian G refers to the fluid particle $\mathbf{r}(t)$ and the Greenian G' to the fluid particle $\mathbf{r}'(t)$. The contribution proportional to λ (not written) can be calculated with the assumption that $\tilde{\omega}$ is purely advected by the large-scale velocity, i.e., $\tilde{\mathbf{u}}\nabla\overline{\omega}$ is neglected in (9). This is the case considered by [15]. However, in this approximation the coarse-grained enstrophy $\int \overline{\omega}^2\, d^2\mathbf{r}$ is conserved [15] and no trend towards a self-organized state (e.g., maximum entropy or minimum enstrophy state) is apparent. The exchange of enstrophy between small and large scales (and also the source of entropy) corresponds to higher order corrections in the equation for $\tilde{\omega}$. In this Letter, we consider exclusively the term of order λ^2 which accounts for a diffusion process, but we do not claim that the term of order λ must be necessarily discarded.

To close the system, it remains for one to evaluate the correlation function $\overline{\tilde{\omega}(\mathbf{r},t)\tilde{\omega}(\mathbf{r}',t)}$. We shall assume that the scale of the kinematic correlations is small with respect to the coarse-graining mesh size and take

$$\overline{\tilde{\omega}(\mathbf{r},t)\tilde{\omega}(\mathbf{r}',t)} = \epsilon^2\delta(\mathbf{r}-\mathbf{r}')\overline{\tilde{\omega}^2}(\mathbf{r},t). \qquad (15)$$

This assumption is consistent with our scale separation hypothesis and was made previously by [6,10] for the Euler equation and by [12–14] in plasma physics. Now

$$\overline{\tilde{\omega}^2} = \overline{(\omega-\overline{\omega})^2} = \overline{\omega^2} - \overline{\omega}^2. \qquad (16)$$

For the case that we consider, the exact vorticity field ω can take only two values, $\omega = 0$ and $\omega = \sigma_0$. This implies that $\overline{\omega^2} = \overline{\sigma_0 \times \omega} = \sigma_0\overline{\omega}$ and, therefore,

$$\overline{\tilde{\omega}(\mathbf{r},t)\tilde{\omega}(\mathbf{r}',t)} = \epsilon^2\delta(\mathbf{r}-\mathbf{r}')\overline{\omega}(\sigma_0-\overline{\omega}). \qquad (17)$$

Substituting this expression in Eq. (14) and carrying out the integration on $\mathbf{r}''$, we obtain

$$\frac{\partial\overline{\omega}}{\partial t} + \overline{\mathbf{u}}\nabla\overline{\omega} = \epsilon^2\frac{\partial}{\partial r^{\mu}}\int_0^t ds\int d^2\mathbf{r}'\, V^{\mu}(\mathbf{r}'\to\mathbf{r})_t \times \Big\{V^{\nu}(\mathbf{r}'\to\mathbf{r})\overline{\omega}'(\sigma_0-\overline{\omega}')\frac{\partial\overline{\omega}}{\partial r^{\nu}} + V^{\nu}(\mathbf{r}\to\mathbf{r}')\overline{\omega}(\sigma_0-\overline{\omega})\frac{\partial\overline{\omega}'}{\partial r'^{\nu}}\Big\}_{t-s}. \qquad (18)$$

We have written $\overline{\omega}'_{t-s} \equiv \overline{\omega}(\mathbf{r}'(t-s),t-s)$, $\overline{\omega}_{t-s} \equiv \overline{\omega}(\mathbf{r}(t-s),t-s)$, $V^{\mu}(\mathbf{r}'\to\mathbf{r})_t \equiv V^{\mu}(\mathbf{r}'(t)\to\mathbf{r}(t))$, and $V^{\nu}(\mathbf{r}'\to\mathbf{r})_{t-s} \equiv V^{\nu}(\mathbf{r}'(t-s)\to\mathbf{r}(t-s))$ where $\mathbf{r}(t-s)$ is the position at time $t-s$ of the fluid particle located in $\mathbf{r}=\mathbf{r}(t)$ at time t. It is determined by the characteristics (12) of the smoothed-out Lagrangian flow.

Equation (18) is a non-Markovian integrodifferential equation: the value of $\overline{\omega}$ in $\mathbf{r}$ at time t depends on the value of the *whole* vorticity field at *earlier times*. If the decorrelation time τ is short, we can make a Markov approximation and simplify the foregoing expression in

$$\frac{\partial\overline{\omega}}{\partial t} + \overline{\mathbf{u}}\nabla\overline{\omega} = \frac{\epsilon^2\tau}{2}\frac{\partial}{\partial r^{\mu}}\int d^2\mathbf{r}'\, V^{\mu}(\mathbf{r}'\to\mathbf{r}) \times \Big\{V^{\nu}(\mathbf{r}'\to\mathbf{r})\overline{\omega}'(\sigma_0-\overline{\omega}')\frac{\partial\overline{\omega}}{\partial r^{\nu}} + V^{\nu}(\mathbf{r}\to\mathbf{r}')\overline{\omega}(\sigma_0-\overline{\omega})\frac{\partial\overline{\omega}'}{\partial r'^{\nu}}\Big\}. \qquad (19)$$

In the case of an infinite domain, $\mathbf{V}(\mathbf{r}\to\mathbf{r}') = -\mathbf{V}(\mathbf{r}'\to\mathbf{r})$ and we have the further simplification

$$\frac{\partial\overline{\omega}}{\partial t} + \overline{\mathbf{u}}\nabla\overline{\omega} = \frac{\epsilon^2\tau}{8\pi^2}\frac{\partial}{\partial r^{\mu}}\int d^2\mathbf{r}'\, K^{\mu\nu}(\mathbf{r}'-\mathbf{r}) \times \Big\{\overline{\omega}'(\sigma_0-\overline{\omega}')\frac{\partial\overline{\omega}}{\partial r^{\nu}} - \overline{\omega}(\sigma_0-\overline{\omega})\frac{\partial\overline{\omega}'}{\partial r'^{\nu}}\Big\}, \qquad (20)$$

where

$$K^{\mu\nu}(\mathbf{r}'-\mathbf{r}) = \frac{\xi_{\perp}^{\mu}\xi_{\perp}^{\nu}}{\xi^4} = \frac{\xi^2\delta^{\mu\nu}-\xi^{\mu}\xi^{\nu}}{\xi^4} \qquad (21)$$

and $\boldsymbol{\xi} = \mathbf{r}'-\mathbf{r}$. The symmetrical form of this equation is of course reminiscent of the generalized Landau equation in plasma physics obtained with a quasilinear theory [12–14].

By introducing a tensor

$$D^{\mu\nu} = \frac{\epsilon^2\tau}{2}\int d^2\mathbf{r}'\, V^{\mu}(\mathbf{r}'\to\mathbf{r})V^{\nu}(\mathbf{r}'\to\mathbf{r}) \times \overline{\omega}'(\sigma_0-\overline{\omega}') \qquad (22)$$

and a vector

$$\eta^{\mu} = \frac{\epsilon^2\tau}{2}\int d^2\mathbf{r}'\, V^{\mu}(\mathbf{r}'\to\mathbf{r})V^{\nu}(\mathbf{r}\to\mathbf{r}')\frac{\partial\overline{\omega}'}{\partial r'^{\nu}}, \qquad (23)$$

Eq. (19) can be rewritten in the more illuminating form

$$\frac{\partial\overline{\omega}}{\partial t} + \overline{\mathbf{u}}\nabla\overline{\omega} = \frac{\partial}{\partial r^{\mu}}\left[D^{\mu\nu}\frac{\partial\overline{\omega}}{\partial r^{\nu}} + \overline{\omega}(\sigma_0-\overline{\omega})\eta^{\mu}\right]. \qquad (24)$$

This equation has the structure of a generalized Fokker-Planck equation with a diffusion term and a drift term. The diffusion term corresponds to the turbulent viscosity introduced *ad hoc* in most parametrizations of turbulence. However, this term alone breaks the conservation laws of the Euler equation. The present theory shows that an additional *drift term* must exist in order to recover these properties. The drift is nonlinear in $\overline{\omega}$ so that (24) is not, strictly speaking, a Fokker-Planck equation. This nonlinearity accounts for the constraint $\overline{\omega}(\mathbf{r}, t) \leq \sigma_0$ imposed at any time by the conservation of the fine-grained vorticity [see Eq. (1)].

Equation (19) respects the invariance properties of the 2D Euler equation and has the same structure as Eq. (23) of Chavanis and Sommeria [11] derived on the basis of thermodynamical arguments. In their Letter, the constraints of the Euler equation were satisfied with the aid of Lagrange multipliers. In this new approach, the conservation laws follow naturally from the symmetrical structure of Eq. (19), as for the usual Landau equation (the linear impulse $P = \int \omega y\, d^2\mathbf{r}$ and the angular mometum $L = \int \omega r^2 d^2\mathbf{r}$ play the role of the impulse $\mathbf{P} = \int f \mathbf{v}\, d^3\mathbf{v}$ and kinetic energy $K = \int f \frac{v^2}{2}\, d^3\mathbf{v}$ in plasma physics). This is more satisfying from a physical point of view. Moreover, in the thermodynamical approach, the increase of entropy is *postulated*, whereas in the present situation an H theorem for the Fermi-Dirac entropy,

$$S = -\int \left\{ \frac{\overline{\omega}}{\sigma_0} \ln \frac{\overline{\omega}}{\sigma_0} + \left(1 - \frac{\overline{\omega}}{\sigma_0}\right) \ln\left(1 - \frac{\overline{\omega}}{\sigma_0}\right) \right\} d^2\mathbf{r}\,, \tag{25}$$

results immediately from Eq. (19). This is proved by taking the time derivative of (25), substituting for (19), interchanging the dummy variables $\mathbf{r}$ and $\mathbf{r}'$, and summing the two resulting expressions. Of course, the increase of entropy is due to the coarse-graining procedure which creates some irreversibility (the indetermination on the position of the vorticity levels in a cell). The entropy $S[\omega]$ for the exact vorticity ω is conserved by the Euler equation as the integral of any function of ω.

It is remarkable that a quasilinear theory is sufficient to generate a turbulent viscosity (but also a drift) and a source of entropy. We do not necessarily have to advocate the nonlinear terms in (7) to get these properties. Note also that the entropy associated with the (coarse-grained) Euler equation is the Fermi-Dirac entropy (25), in agreement with the works of [3,4] at equilibrium. Unfortunately, Eq. (19) does not conserve energy exactly. Therefore, the system will ultimately relax towards the solution $\overline{\omega} = \sigma_0/[1 + \lambda \exp(\alpha \sigma_0 r^2)]$ which is the maximum entropy state at fixed circulation and angular momentum. This means that our approximations break down at very late times.

A further connection with the statistical theory of 2D turbulence can be found. Equation (19) is an integrodifferential equation, whereas the equations derived from the MEPP [6,9,11] are differential equations. The usual way to transform an integrodifferential equation into a differential equation is to make a guess for the function $\overline{\omega}'$ appearing in the integral. It makes sense to replace $\overline{\omega}'$ by its optimal value $\overline{\omega}' = \sigma_0/[1 + \lambda \exp(\beta \sigma_0 \psi')]$, maximizing entropy at fixed energy and circulation. By substituting in (22) and (23) and making a "local approximation," we obtain

$$\boldsymbol{\eta} = D \beta \nabla \psi\,, \tag{26}$$

$$D = \frac{\tau \epsilon^2}{8\pi} \ln\left(\frac{L}{\epsilon}\right) \overline{\omega}(\sigma_0 - \overline{\omega})\,. \tag{27}$$

In Eq. (26), we recover the form of the drift derived by Chavanis [17] in a point vortex model. The drift coefficient can be interpreted as an Einstein formula. By substituting for the drift in (24), we recover the equation

$$\frac{\partial \overline{\omega}}{\partial t} + \overline{\mathbf{u}} \nabla \overline{\omega} = \nabla (D [\nabla \overline{\omega} + \beta \overline{\omega}(\sigma_0 - \overline{\omega}) \nabla \psi]) \tag{28}$$

obtained by Robert and Sommeria [9] using a maximum entropy production principle. Equation (28) can be interpreted as a generalized Fokker-Planck equation [17]. Note that the present approach provides the value (27) of the diffusion coefficient which was left unknown by the variational principle [9]. This value coincides with the estimate of [6,10] based on a passive scalar model.

The results of this Letter can be extended to an arbitrary spectrum of vorticity levels [8] for a wider class of initial conditions. These results also complete the analogy between 2D turbulence and stellar systems investigated by the author [6,7].

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