Quantum Transport through Ballistic Cavities: Soft vs Hard Quantum Chaos

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We study transport through a two-dimensional billiard attached to two infinite leads by numerically calculating the Landauer conductance and the Wigner time delay. In the generic case of a mixed phase space we find a power-law distribution of resonance widths and a power-law dependence of conductance increments apparently reflecting the classical dwell time exponent, in striking difference to the case of a fully chaotic phase space. Surprisingly, these power laws appear on energy scales *below* the mean level spacing, in contrast to semiclassical expectations.

Advances in the fabrication of semiconductor heterostructures and metal films have made it possible to produce two-dimensional nanostructures with a very low amount of disorder [1]. At low temperatures, scattering of the electrons happens mostly at edges of the structures with the electrons moving ballistically between collisions with the boundary. Theoretical and experimental investigations have shown that the spectral and transport properties of such quantum coherent cavities, commonly called "billiards," depend strongly on the nature of their classical dynamics. In particular, integrable and chaotic systems were found to behave quite differently [2,3].

Generic billiards are neither integrable nor ergodic [4], but have a mixed phase space with regions of regular as well as chaotic dynamics [5]. Their dynamics is much richer than in either of the extreme cases, as phase space has a hierarchical structure at the boundary of regular and chaotic motion. In particular, this leads to a trapping of chaotic trajectories close to regular regions with a probability $P(t) \sim t^{-\beta}$ for $t > t_0$, to be trapped longer than a time t, with t_0 of the order of a few traversal times [6]. The exponent $\beta > 1$ depends on system and parameters with typically $\beta \approx 1.5$ [6]. This power-law trapping in mixed systems is in contrast to the typical exponentially decaying staying probability of fully chaotic systems (see Fig. 1).

Recently, it was shown semiclassically employing the diagonal approximation that the variance of conductance increments (for a small dc bias voltage) over small energy intervals ΔE grows as [7,8]

$$\Delta g^2(\Delta E) \equiv \langle [g(E + \Delta E) - g(E)]^2 \rangle_E \sim |\Delta E|^\beta, \quad (1)$$

for mixed systems if $\beta < 2$. This is in strong contrast to an increase as $(\Delta E)^2$ in the case of fully chaotic systems [3]. The semiclassical approximation requires ΔE to be *larger* than the mean level spacing Δ, corresponding to the picture that quantum mechanics can follow the classical power-law trapping at most until the Heisenberg time $t_H = h/\Delta$ [9].

In the semiclassical approximation the graph of g vs E has the statistical properties of fractional Brownian motion with a fractal dimension $D = 2 - \beta/2$ [7]. Fractal conductance fluctuations have indeed been found in experiments on gold wires [10] and semiconductor nanostructures [11] and numerically for the quantum separatrix map [12].

In this Letter, we numerically study quantum transport through a simple cavity, the cosine billiard [13] (see insets of Fig. 1). Although we observe completely different behavior for the mixed and fully chaotic cases in the semiclassical regime of many (45) transmitting modes, we find in the mixed case no indication of fractality or fractional Brownian motion behavior of the graph g vs E. This is the first surprise, as it is in contrast to the above mentioned semiclassical [7], experimental [10,11], and numerical [12]
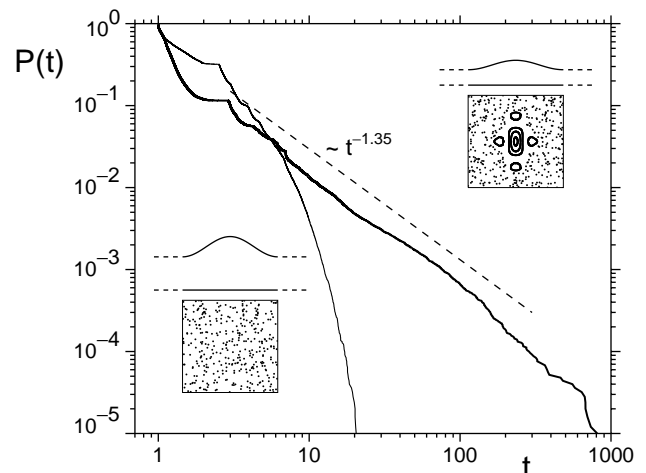

FIG. 1. Classical dwell time probability $P(t)$, for the cases of mixed (thick line) and fully chaotic (thin line) dynamics with t in units of the traversal time. The insets show the shape of the billiard with attached leads and a Poincaré surface of section of the closed billiard for the mixed (upper right) and chaotic (lower left) case.

works. Instead, the conductance is characterized by narrow isolated resonances, with the classical exponent β appearing in a power-law distribution of resonance widths *smaller* than the mean level spacing. This leads to a scaling of $\Delta g^2(\Delta E)$ in agreement with the semiclassically derived Eq. (1), however, only on scales *below* the mean level spacing. This surprising result contradicts the semiclassical intuition that quantum mechanics may mimic classical properties at most until the Heisenberg time corresponding to energy scales *above* the mean level spacing. At present, there is no explanation for these numerical results. They show that even with a detailed (semiclassical) knowledge of the universal chaotic regime as well as the integrable case at hand we are just at the beginning of understanding the quantum properties of *generic* Hamiltonian systems.

The cosine billiard [13] is defined by two hard walls at $y = 0$ and $y(x) = W + (M/2)[1 - \cos(2\pi x/L)]$, for $0 \leq x \leq L$, with two semi-infinite perfect leads of width W attached to the openings of the billiard at $x = 0$ and $x = L$ (see insets of Fig. 1). By changing the parameter ratios W/L and M/L the stability of periodic orbits associated with the billiard can be changed, allowing a transition from a mixed to a predominantly chaotic phase space. Note that in the mixed case the leads couple to the chaotic part of phase space only.

The S matrix of the system has been calculated by the recursive Green's function method after expanding the two-dimensional wave function in terms of local transverse energy eigenfunctions [14]. In the numerical calculations, it was checked that a sufficient number of modes in the expansion in transverse eigenmodes was kept and that the lattice constant in the x direction was sufficiently small. For a given energy $E_F = \hbar^2 k_F^2/2m$, N modes in the leads are transmitting, with $k_F W/\pi \gtrsim N$. We turn from the S matrix to the experimentally relevant conductance at small dc bias voltage using the Landauer formula, $G = e^2/h \, \text{Tr}(tt^\dagger)$, where t is the transmission matrix. Spectral information is contained in the Wigner-Smith time delay $\tau = -i\hbar \, \text{Tr}(S^\dagger dS/dE)/2N$, where $2N$ is the dimension of the S matrix. All energies in this paper are given in units of $\hbar^2\pi^2/(2mW^2)$.

Figure 2 shows the dimensionless conductance $g = G(h/e^2)$ and the Wigner-Smith time delay τ [in units of $2mW^2/(\hbar\pi^2)$] for parameters corresponding to a mixed phase space ($W/L = 0.18$, $M/L = 0.11$) and a chaotic phase space with no apparent stability island ($W/L = 0.36$, $M/L = 0.22$) for $N = 45$ transmitting modes. The differences are quite dramatic. For the fully chaotic case, both quantities are smooth functions of energy and in good agreement with semiclassical theory (see below). While the average values are comparable, many sharp isolated resonances on top of a smooth background are visible in the mixed case [15], also in contrast to the semiclassically predicted fractional Brownian motion. The simple explanation that these narrow resonances are related to quantum tunneling into the islands of regular
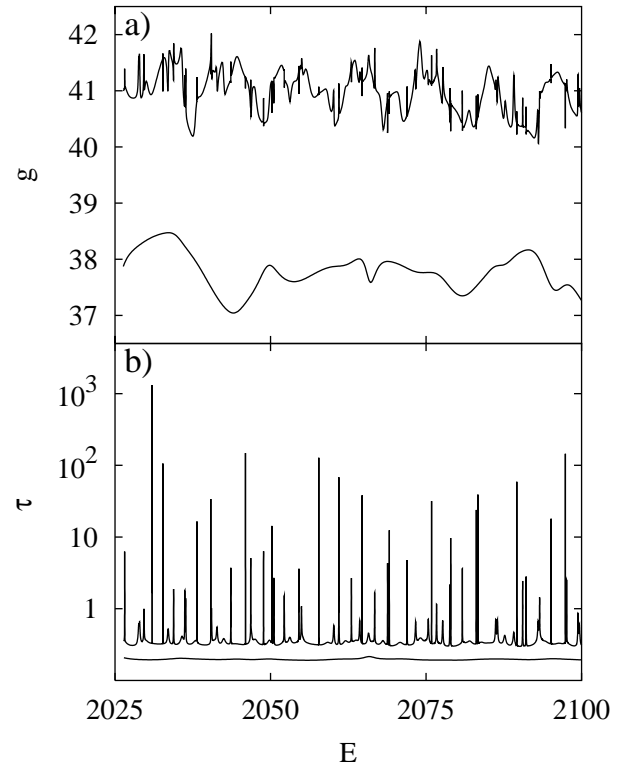

FIG. 2. Dimensionless conductance g (a) and Wigner time delay τ (b) for $N = 45$ propagating modes. Thick (thin) lines for mixed (fully chaotic) case.

motion [16] does not apply here, as the phase space volume of stable islands is about 5%, while the narrow resonances (below the mean level spacing) make up about 18% of all states associated with the billiard. This roughly corresponds to the phase space volume around the stable islands where trapping of chaotic trajectories occurs.

In order to analyze the narrow resonances in the mixed case, it is convenient to examine the Wigner-Smith time delay. Each resonance in the time delay has the Breit-Wigner shape, characterized by a width Γ_i and a height τ_i situated at an energy E_i on top of a smooth background. We find our data well described by

$$\tau(E) = \sum_i \tau_i \frac{\Gamma_i^2/4}{(E - E_i)^2 + \Gamma_i^2/4} + \tau_{\text{smooth}}(E), \quad (2)$$

with $\tau_{\text{smooth}}(E) \propto E^{-1/2}$. Since the phase shift through a resonance is 2π, width and height are related by $\tau_i\Gamma_i = 2/N$. The energy was initially sampled on an equidistant grid and subsequently refined in order to resolve the sharp resonances. Only resonances with a $\Gamma \lesssim 10^{-3}$, i.e., much smaller than the initial grid are lost. As a result we can numerically construct the cumulative distribution $N(\Gamma)$ of resonance widths, corresponding to the probability of finding a resonance smaller than Γ (Fig. 3). The distribution is very broad, spanning 5 orders of magnitude, and is
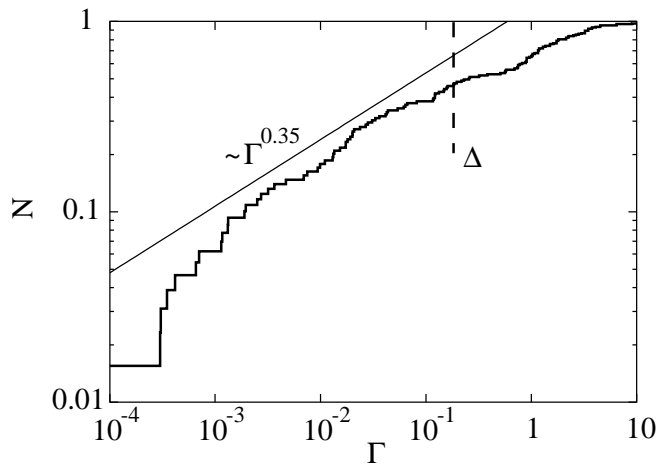
FIG. 3. Normalized cumulative distribution of resonance widths $N(\Gamma)$ for the mixed case. The thin line serves as a guide to the eye. The mean level spacing Δ is shown in the figure.

approximately a power law $N(\Gamma) \approx a\Gamma^r$, with $r \approx 0.35$, over a wide range *below* the mean level spacing $\Delta = 0.176$.

The consequences of this broad distribution of resonance widths for the variances of conductance and time delay increments are studied now. For $\Delta E \ll \Delta$ correlations between different isolated resonances ($\Gamma \ll \Delta$) do not contribute to the variance Δg^2, which then is governed by the distribution of resonance widths. Each resonance is reflected in the conductance,

$$g(E) = g_{\text{smooth}}(E) + \sum_{i=1}^{N_R} \delta g_i(E), \qquad (3)$$

where $\delta g_i(E)$ is a function of the width Γ_i and the typical height $\tilde{g}_i$. N_R is the number of resonances with $\Gamma_i < \Delta$ in the energy interval $E_B - E_A$ over which we take the average. The variance of the increments of a single resonance is given by

$$\langle [\delta g_i(E + \Delta E) - \delta g_i(E)]^2 \rangle_E = \frac{\tilde{g}_i^2 \Gamma_i}{E_B - E_A} \begin{cases} b_i(\Delta E/\Gamma_i)^2, & \Delta E \ll \Gamma_i, \\ 1, & \Delta E \gg \Gamma_i, \end{cases} \qquad (4)$$

which defines $\tilde{g}_i$ and where b_i is a numerical factor of order unity. Since the distribution of widths is very broad, the strong inequalities are almost always fulfilled in the sum over resonances. Splitting this sum into resonances smaller and larger than ΔE, we get

$$\Delta g^2(\Delta E) \approx \frac{1}{E_B - E_A} \left(\sum_{\Gamma_i < \Delta E} \tilde{g}_i^2 \Gamma_i + \sum_{\Gamma_i > \Delta E} b_i \tilde{g}_i^2 \frac{(\Delta E)^2}{\Gamma_i} \right). \qquad (5)$$

Replacing the sums by integrals over the density of widths $n(\Gamma) \approx ar\Gamma^{r-1}$ and neglecting the weak fluctuations of $\tilde{g}_i$ and b_i as compared to Γ_i, we can estimate for small ΔE,

$$\Delta g^2(\Delta E) \propto \langle \tilde{g}^2 \rangle \frac{N_R}{E_B - E_A} a |\Delta E|^{1+r}, \qquad (6)$$

where $\langle \cdots \rangle$ stands for the average over isolated resonances. A power-law distribution of resonances thus leads to a power-law increase of the variance of conductance increments with the exponent given by $1 + r$.

Figure 4(a) shows the variances of the conductance increments. On scales smaller than the minimum resonance width the variance increases quadratically, as expected. On larger scales we find the power law Eq. (6). This result coincides with the semiclassically derived Eq. (1) with $r = \beta - 1$, however, only on scales *below* the mean level spacing. At present, there is no explanation why the classical exponent β appears on such small energy scales. Remarkably, on scales above the mean level spacing the correlation energy for the conductance fluctuations is given by the Weisskopf width $\Gamma_W \approx 2$, as in the fully chaotic case (see below).

The variance of increments of the time delay are shown in Fig. 4(b). Since the variance $\Delta \tau^2(\Delta E)$ measures the square of the resonance peak height in the time delay, in the mixed case, they are completely dominated by the sharpest resonance, once the energy exceeds the minimum resonance width. Thus, in contrast to fully chaotic systems, in mixed systems the scale of the correlations of the time delay is the smallest resonance width and not the Weisskopf width Γ_W.

For comparison in Fig. 4 we also show the results for $\Delta g^2(\Delta E)$ and $\Delta \tau^2(\Delta E)$ for the fully chaotic case. They are characterized by single scales $\Gamma_g \approx 4.8$ and $\Gamma_\tau \approx 3.8$ and are in good agreement with semiclassical results [3,18]. The chaotic case can also be described by the random matrix theory (RMT) [19], whose results coincide with the cited semiclassical ones for $N \gg 1$ [20]. In the absence of direct processes, RMT predicts a single correlation scale, known as Weisskopf correlation width, $\Gamma_W = \Delta/2\pi \sum_c T_c$, where the sum runs over all channels c with transmission probability T_c [21]. Approximating $\sum_c T_c$ by twice the average dimensionless conductance we obtain $\Gamma_W \approx 4.2$, in agreement with the numerical values within the statistical accuracy. Before concluding, it is worthwhile to stress that, depending on N and the coupling to the leads, quantum chaotic scattering can also exhibit isolated resonances. Their width distribution, however, follows a χ^2 distribution with N degrees of freedom [19], rather than power law.

In conclusion, we have shown that generic Hamiltonian systems, which have regular as well as chaotic phase space regions, differ drastically in the Landauer conductance and Wigner time delay from fully chaotic systems. We find many isolated narrow resonances with
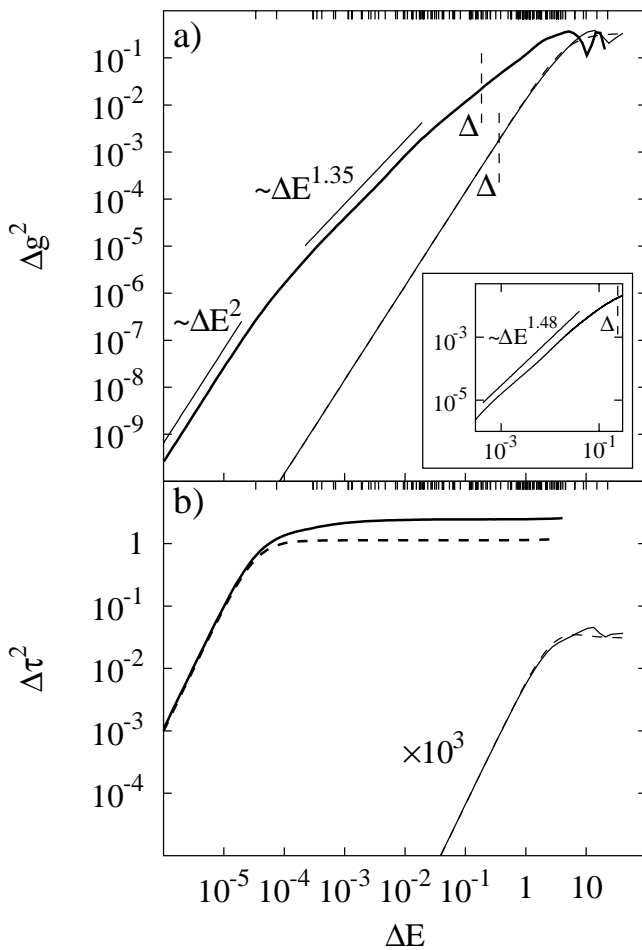
FIG. 4. Variance of the increments of the conductance $\Delta g^2(\Delta E)$ (a) and Wigner time delay $\Delta \tau^2(\Delta E)$ (b). The thick (thin) lines correspond to the mixed (fully chaotic) case. The thin dashed lines are fits to the semiclassical expressions for the chaotic case. The thick dashed line in (b) is the contribution of the sharpest resonance to the variance for the mixed case. For comparison, the ticks on the upper borders mark the widths of the individual resonances. The inset in (a) shows the corresponding data for a quantum graph modeling a mixed phase space (classical $\beta = 1.48$) and was provided by the authors of Ref. [17]. The mean level spacing Δ is shown in the figures.

a power-law distribution of their widths accompanied by a power-law increase of the variance of conductance increments. Both power laws appear to be connected to the classical power-law trapping, surprisingly they appear only on scales *below* the mean level spacing. Similar unexplained power laws are found in recent studies using quantum graphs [22] modeling a mixed phase space [see inset of Fig. 4(a)] [17]. Further research on the quantum signatures of classically mixed systems is urgently needed.

Note added.—The authors of Ref. [17] have informed us that there are quantum graphs where the classical and quantum exponents do not agree.

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