C, P, and T Invariance of Noncommutative Gauge Theories

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In this paper we study the invariance of the noncommutative gauge theories under C, P, and T transformations. For the noncommutative space (when only the spatial part of θ is nonzero) we show that noncommutative QED (NCQED) is parity invariant. In addition, we show that under charge conjugation the theory on noncommutative R^4_θ is transformed to the theory on $R^4_{-\theta}$, so NCQED is a CP violating theory. The theory remains invariant under time reversal if, together with proper changes in fields, we also change θ by $-\theta$. Hence altogether NCQED is CPT invariant. Moreover, we show that the CPT invariance holds for general noncommutative space-time.

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I. INTRODUCTION

Recently it has been shown that the noncommutative spaces arise naturally when one studies the perturbative string theory in the presence of D branes with nonzero B field background, i.e., the low energy world volume theory on such branes is a noncommutative supersymmetric gauge theory (for a review of the field, see [1]).

Besides the string theory arguments, the noncommutative field theories by themselves are very interesting. Generally, the noncommutative version of a field theory is obtained by replacing the product of the fields appearing in the action by the $*$-product:

$$(f * g)(x) = \exp\left(\frac{i}{2}\,\theta_{\mu\nu}\partial_{x\mu}\partial_{y\nu}\right)f(x)g(y)\,|_{x=y}\,, \quad (1)$$

where f and g are two arbitrary functions, which we assume to be infinitely differentiable. The "Moyal Bracket" of two functions is

$$\{f, g\}_{\mathrm{M.B.}} = f * g - g * f\,.$$

It is apparent that if we choose f and g to be the coordinates themselves we find

$$\{x_\mu, x_\nu\} = i\theta_{\mu\nu}\,, \quad (2)$$

and this is why these spaces are called noncommutative. Moreover, consistently we assume the derivatives to act trivially on this space:

$$\{x_\mu, \partial_\nu\} = -\eta_{\mu\nu} \qquad \{\partial_\mu, \partial_\nu\} = 0\,. \quad (3)$$

Because of the nature of the $*$-product, the noncommutative field theories for the slowly varying fields or low energies ($\theta E^2 \lesssim 1$), at *classical* level, effectively reduce to their commutative version. However, this is only the classical result and quantum corrections will show the effects of θ even at low energies [2]. Since the derivatives are commuting after rewriting the noncommutative fields and their action in terms of the Fourier modes we find a commutative field theory in the momentum space, and this field theory has unfamiliar momentum dependent interactions [3]. In this way, we find a tool to study these theories perturbatively, like the usual commutative field theories.

It has been shown that the noncommutative version of ϕ^4 theory in four dimensions is two loop renormalizable [2,4]; moreover, it is shown that the noncommutativity parameter, θ, *does not receive quantum corrections*.

The pure noncommutative U(1) theory has been discussed and shown to be one loop renormalizable. The one loop beta function for noncommutative U(1) is negative (and hence the theory is asymptotically free). The interesting result is that this one loop beta function is *not θ dependent* [3,5,6]. However, it is not clear whether this property remains at higher loops. It is believed that all of these one loop properties are a consequence of the fact that the planar degrees of freedom of noncommutative theories is the same as a commutative theory [7]. The question of the renormalizability has also been addressed for noncommutative QED [noncommutative U(1)+ fermions] [8].

In this paper we study another interesting question about noncommutative theories regarding their behavior under discrete symmetries. Since in the noncommutative spaces we have missed the Lorentz symmetry, discrete symmetries and in particular the CPT invariance, in the context of noncommutative geometry in general, are usually nontrivial questions. This question has been very briefly discussed in [8]. Hence, first we should build the noncommutative version of QED, NCQED. We show that there are two distinct choices for the fermion representations. We will show that these two are related by charge conjugation, so we may call them positively or negatively charged representations. We will give more intuitive explanations for these representations. In Sec. III, we study the behavior of our theory under discrete symmetries. In this section we show the explicit calculations for the cases with $\theta_{0i} = 0$ (x^0 is the time coordinate) and we present only the results for the nonzero θ_{0i} case in the last part of this section. For the $\theta_{0i} = 0$ cases, we show that our theory, NCQED, is parity invariant, with the usual transformation of the fields; and studying the charge conjugation transformations we show that the NCQED is not C invariant and in order to make the theory invariant besides the usual field transformations we should also change θ by $-\theta$. In addition, we show that the same θ changing is needed for time reversal invariance. So, although our theory is CP violating, it is CPT invariant. For *the general* θ we show that though C, P, and T are broken, the whole theory is again CPT invariant. The last section is devoted to conclusions and remarks.

II. BUILDING THE NCQED

(i) Pure gauge theory.—The action for the pure gauge theory is

$$S = \frac{1}{4\pi} \int F_{\mu\nu} * F^{\mu\nu}\, d^4x = \frac{1}{4\pi} \int F_{\mu\nu} F^{\mu\nu}\, d^4x\,, \tag{4}$$

with

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + ig\{A_\mu, A_\nu\}\,. \tag{5}$$

In the above g is the gauge coupling constant. Let us consider the following transformations

$$\begin{aligned} A_\mu \to A'_\mu &= U(x) * A_\mu * U^{-1}(x) \\ &\quad + \tfrac{i}{g} U(x) * \partial_\mu U^{-1}(x)\,, \\ U(x) = \exp * (i\lambda), &\qquad U^{-1}(x) = \exp * (-i\lambda)\,, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \exp * (i\lambda(x)) &\equiv 1 + i\lambda - \tfrac{1}{2}\lambda * \lambda \\ &\quad - \tfrac{i}{3!}\lambda * \lambda * \lambda + \cdots \\ U(x) * U^{-1}(x) &= 1\,. \end{aligned} \tag{7}$$

Under the above transformations

$$F_{\mu\nu} \to F'_{\mu\nu} = U(x) * F_{\mu\nu} * U^{-1}(x)\,. \tag{8}$$

Hence due to the cyclic property of integration over space (see Appendix A) the action is invariant under (6). The above argument can be trivially generalized to U(N) cases by letting A and λ take values in the related algebra. However, in this paper we consider only the U(1) case.

(ii) Fermionic part.—In order to write down the fermionic part of the action with the noncommutative U(1) symmetry mentioned above, first we need to find the proper "fundamental" representations of the noncommutative U(1) group. There are two distinct choices for that: (a) The representation with

$$\begin{cases} \psi_+(x) \to \psi'_+(x) = U(x) * \psi_+(x)\,, \\ \overline{\psi}_+(x) \to \overline{\psi}'_+(x) = \overline{\psi}_+(x) * U^{-1}(x)\,, \end{cases} \tag{9}$$

and (b) the other with

$$\begin{cases} \psi_-(x) \to \psi'_-(x) = \psi_-(x) * U^{-1}(x)\,, \\ \overline{\psi}_-(x) \to \overline{\psi}'_-(x) = U(x) * \overline{\psi}_-(x)\,. \end{cases} \tag{10}$$

We will show that these two types of fermions are related by a charge conjugation transformation. The next step is finding a "covariant" derivative, D_μ. For these two types of fermions we have different "covariant" derivatives: (a)

$$D^\mu_+ \psi_+(x) \equiv \partial^\mu \psi_+(x) - ig\psi_+(x) * A^\mu(x)\,, \tag{11}$$

and (b)

$$D^\mu_- \psi_-(x) = \partial^\mu \psi_-(x) + igA^\mu(x) * \psi_-(x)\,. \tag{12}$$

One can show that with each of the covariant derivatives defined above (with our proper fermionic representation), the action

$$S = \int d^4x\, \overline{\psi} * (i\gamma^\mu D_\mu \psi - m\psi) \tag{13}$$

is invariant under the noncommutative U(1) transformations.

III. *P*, *C*, AND *T* INVARIANCE

Having the form of the action we are ready to study the P, C, and T symmetries. For the sake of certainty up to the last part of this section we consider the noncommutative spaces, i.e., $\theta_{0i} = 0$, and in the last paragraph we discuss the nonzero θ_{0i} and the most general noncommutative space-time.

Parity.—Under the parity, $x_i \to -x_i$, the θ parameter is not changed [see (2)]. It is straightforward to show that for parity transformation given by

$$\begin{cases} A_0 \to A_0\,, \\ A_i \to -A_i\,, \\ \psi(x) \to \gamma^0 \psi\,, \\ x_i \to -x_i\,, \end{cases} \tag{14}$$

the NCQED action is invariant for both of the fermionic choices.

Charge conjugation.—Let us first study the pure noncommutative U(1) case. Under the usual charge conjugation, C-, transformations,

$$A_\mu \to -A_\mu\,, \tag{15}$$

(4) is not invariant, because the first term in the F will change the sign but the second term, $\{A_\mu, A_\nu\}$, will remain the same. To make the theory C invariant we note that the Moyal bracket changes the sign if together with (15) we also change θ by

$$\theta \to -\theta\,. \tag{16}$$

The above θ transformation has an intuitive explanation. As discussed in [3,9], the gauge symmetry of noncommutative U(1) is an infinite dimensional algebra which its Cartan subalgebra (the zero momentum sector) is a U(1) leading to a photonlike state, and all the other gauge particles look like *dipoles* under this U(1), whose dipole moment is proportional to the θ. So, in this picture we feel the necessity of (16), under charge conjugation.

Hence, the noncommutative U(1) theory with parameter θ is mapped into another noncommutative U(1) theory with $-\theta$.

Now, we should consider the fermionic part. Since the kinetic part of the fermionic action is unchanged, we take the usual C transformations

$$\begin{cases} \psi \to i\gamma^0\gamma^2\overline{\psi}^T = -i\gamma^2\psi^*, \\ \overline{\psi} \to i\psi^T\gamma^2\gamma^0. \end{cases} \tag{17}$$

Let us first discuss the fermions in + representation [type (a) fermions]. Under the above transformation, *without changing θ*

$$\int d^4x\, \overline{\psi} * [i\gamma^\mu A_\mu(x) * \psi] \to \int d^4x\, \overline{\psi} * [i\gamma^\mu \psi * A_\mu(x)], \tag{18}$$

we see that this is exactly the form of interaction term for the type (b) fermions. In other words, the types (a) and (b) fermions are charge conjugate of each other. Let us consider the θ transformation too. By using roles given in Appendix A, we see that forms of the interaction term for these two types of fermions are related by (16), which means that (17) together with (15) and (16) give proper charge conjugation transformations, a discrete symmetry of NCQED.

Time reversal.—First we consider the pure noncommutative U(1) and then we study fermions. Under the time reversal, in order to keep the kinetic part of our gauge field action, A_μ should transform as

$$\begin{cases} A_0 \to A_0 \\ A_i \to -A_i. \end{cases} \tag{19}$$

Now let us look at the terms with Moyal brackets. Since time reversal operator involves a complex conjugation, for any two real fields, f and g we have

$$f * g|_\theta \to f_T * g_T|_{-\theta} = g_T * f_T|_\theta, \tag{20}$$

where f_T and g_T show the time reversed f and g, respectively, then we have

$$\{f, g\}_{\text{M.B.}} \to -\{f_T, g_T\}_{\text{M.B.}}. \tag{21}$$

Since A_μ's are real fields,

$$ig\{A^\mu, A^\nu\} \to ig\{A_T^\mu, A_T^\nu\}, \tag{22}$$

and

$$\begin{aligned} F_{0i} &= \partial_{[0}A_{i]} + ig\{A_0, A_i\} \to \partial_{[0}A_{i]} - ig\{A_0, A_i\}, \\ F_{ij} &= \partial_{[i}A_{j]} + ig\{A_i, A_j\} \to F_{ij} = \partial_{[i}A_{j]} - ig\{A_i, A_i\}, \end{aligned} \tag{23}$$

the only way to make the theory invariant under time reversal is changing θ as well as A_μ:

$$\theta \to -\theta. \tag{24}$$

So (19) together with (24) give the proper time reversal transformations.

The Fermionic part.—Since the kinetic term is quadratic in fields, ψ's should obey the usual time reversal transformations:

$$\begin{cases} \psi \to i\gamma^1\gamma^3\psi, \\ \overline{\psi} \to i\overline{\psi}\gamma^1\gamma^3. \end{cases} \tag{25}$$

As for the interaction term, for the sake of certainty let us consider the type (a) case, *without changing θ*. We find

$$\int d^4x\, \overline{\psi} * [i\gamma^\mu A_\mu(x) * \psi] \to \int d^4x\, \overline{\psi}_T * [i\gamma^{*\mu} A_\mu^T(x) * \psi_T]|_{-\theta}, \tag{26}$$

where $\gamma^{*\mu}$ is the complex conjugate of γ^μ. Replacing ψ_T and A_T from (19) and (25), we obtain

$$\int d^4x\, \overline{\psi}_T * [i\gamma^{*\mu} A_\mu^T(x) * \psi_T]|_{-\theta} = -\int d^4x\, \overline{\psi} * [i\gamma^\mu A_\mu(x) * \psi]|_{-\theta}, \tag{27}$$

which is exactly the interaction term for type (b) fermions. As we see, in order to make the NCQED time reversal invariant, we should consider (19), (24), and (25) together.

CPT.—Now that we have studied P, C, and T, it is interesting to consider the CP and CPT too. As we showed, parity transformations remain the same as the commutative version, however each of C and T involves an extra $\theta \to -\theta$. So altogether the NCQED (with parameter θ) is CP violating, i.e., it maps the theory into NCQED with $-\theta$, and the theory is CPT invariant. We should note that although our system is not manifestly Lorentz invariant CPT, as an accidental symmetry, remains valid.

Nonzero θ_{0i} and general $\theta_{\mu\nu}$.—Although a well-defined Hamiltonian for nonzero θ_{0i} cases is not found yet and hence the quantum theory for these cases is not understood in the same sense as θ_{ij} case, one can formally study the discrete symmetries for these cases. As it is readily seen from (2) under parity the θ_{0i} components should be replaced with $-\theta_{0i}$, and we can show that the (14) transformations together with this θ change is the symmetry of NCQED.

For the charge conjugation to make the theory invariant the change in θ parameter, (16), should be extended to θ_{0i} components too.

It is straightforward to check that the time reversal invariance is achieved if θ_{0i} are unchanged while the θ_{ij} components should be transformed by (24). Keeping in mind that time reversal involves a complex conjugation, this result is expected from (2). Hence for general $\theta_{\mu\nu}$ the theory remains CPT invariant, although the theory violates P, C, and T.

IV. CONCLUSIONS AND REMARKS

In this paper we have reviewed the noncommutative gauge theory and their gauge symmetry and shown that fermions can be added in two distinct fundamental representations of the gauge group. We have shown that these two representations are related by charge conjugation, so we called them positive or negative representations.

Studying the discrete symmetries for the $\theta_{0i} = 0$ cases, we have shown that NCQED is parity invariant under the usual (commutative) field transformations. For C and T transformations we showed that besides the usual field transformations we need an extra $\theta \to -\theta$ transformation. In other words, NCQED with θ is charge conjugated (or time reversed) of NCQED with $-\theta$. Therefore, despite being Lorentz noninvariant, in this case NCQED is CT invariant, and hence CPT invariant. In other words P and CPT is an accidental symmetry of the system.

For the general $\theta_{\mu\nu}$, we discussed that P, C, and T invariance are broken; however, the theory is again CPT invariant.

Noncommutative gauge theories seem to provide a very good framework for the *CP violating* models, which are of great importance in particle physics phenomenology. The advantage of these theories is that the beta function is not θ dependent and futhermore θ does not receive quantum corrections. Therefore the amount of CP violation is completely under control.

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Appendix: Some useful identities in $*$-product calculus.—Let f, g be two arbitrary functions on noncommutative R^d:

$$f(x) = \int f(k) e^{ik\cdot x}\, d^d k, \qquad g(x) = \int g(k) e^{ik\cdot x}\, d^d k\,.$$

Then

$$(f * g)(x) = \int f(k) g(l) e^{-ik\theta l/2} e^{i(k+l)\cdot x}\, d^d k\, d^d l\,,$$

where $k\theta l = k^\mu \theta_{\mu\nu} l^\nu$. From the above relation it is straightforward to see: (1) $g * f = f * g|_{\theta\to-\theta}$, and hence $\{f, g\}_{\mathrm{M.B.}} = f * g|_\theta - f * g|_{-\theta}$. (2) $\int (f * g) \times (x)\, d^d x = \int (g * f)(x)\, d^d x = \int fg(x)\, d^d x$. (3) If we denote complex conjugation by c.c., then $(f * g)^{\mathrm{c.c.}} = g^{\mathrm{c.c.}} * f^{\mathrm{c.c.}}$. If h is another arbitrary function: (4) $(f * g) * h = f * (g * h) \equiv f * g * h$. (5) $\int (f * g * h)(x)\, d^d x = \int (h * f * g)(x)\, d^d x = \int (g * h * f) \times (x)\, d^d x$. (6) $(f * g * h)|_\theta = (h * g * f)|_{-\theta}$.

In other words the integration on the space coordinates, x, has the cyclic property, and it has all the properties of the Tr in the matrix calculus.

From (2) we learn that the kinetic part of the actions (which are quadratic in fields) is the same as their commutative version. So the free field propagators in commutative and noncommutative spaces are the same.

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