Quantum Magnetic Collapse

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We study the thermodynamics of degenerate electron and charged vector boson gases in very intense magnetic fields. In degenerate conditions of the electron gas, the pressure transverse to the magnetic field B may vanish, leading to a transverse collapse. For W bosons an instability arises because the magnetization diverges at the critical field $B_c = M_W^2/e$. If the magnetic field is self-consistently maintained, the maximum value it can take is of the order of $2B_c/3$, but in any case the system becomes unstable and collapses.

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Large magnetic fields can be generated due to gravitational and rotational effects in stellar objects like supernovas and neutron stars; i.e., magnetic fields of order 10^{20} G and larger have been suggested to exist in the cores of neutron stars [1]. The standard electroweak theory establishes a limit on the magnetic field, the critical upper bound for stable vacua being $B_c = M_W^2/e \simeq 1.06 \times 10^{24}$ G, coming from the $W^\pm$ ground state energy $\epsilon_{0q} = \sqrt{M_W^2 - eB}$, which is imaginary for $B > B_c$. Fields of order B_c may have been created at the electroweak phase transition (see [2,3]). The galactic and intergalactic magnetic fields can be considered as relics of such huge magnetic fields in the early Universe [4–8]. In astrophysics, also the critical field $B_{c'} = m_e^2/e \simeq 4.41 \times 10^{13}$ G is relevant.

Nielsen, Olesen, and Ambjørn [9,10] showed that the vacuum possesses the properties of a ferromagnet or an antiscreening superconductor for $B \sim B_c$. It thus seems relevant to study the electroweak medium in a strong magnetic field of the order of the critical magnetic fields. The implications of these results for astroparticle physics and cosmology are expected to be interesting. As in preceding papers (Refs. [11,12]), we consider only the first generation of leptons and quarks for the sake of simplicity. Here we calculate the magnetization due to the charged leptons and intermediate vector bosons in the standard model.

The thermodynamic potential $\Omega = -T \ln \mathcal{Z}$ involves contributions from leptons and quarks, which are considered to be in chemical equilibrium among themselves through the boson fields, described by equations among their chemical potentials [11] like $\mu_{W^+} = \mu_\nu + \mu_{e^+}$, $\mu_{d_L} + \mu_{W^+} = \mu_{u_L}$, $\mu_{e^+,W^+} + \mu_{e^-,W^-} = 0$. From the thermodynamical potential we choose the electron and W sectors exhibiting interesting effects in the astrophysical and cosmological scenarios, respectively, in the presence of extremely strong magnetic fields ($B \sim B_{c'}$ and $B \sim B_c$).

In the astrophysical scenario the electron-positron gas thermodynamics is of interest. In the cosmological context, we will be concerned especially with the $W^\pm$ sector.

The one-loop thermodynamical potential per unit volume of the electron-positron sector is $\Omega_e = \Omega_{se} + \Omega_{0e}$, where

$$\Omega_{se} = -\frac{eB}{4\pi^2\beta}\sum_{n=0}^{\infty} a_n \times \int_{-\infty}^{\infty} dp_3 \ln[(1 + e^{-(E_q-\mu_e)\beta})(1 + e^{-(E_q+\mu_e)\beta})]. \quad (1)$$

Here the sum extends over all Landau quantum numbers and the degeneracy factor is $a_n = 2 - \delta_{0n}$, $E_q = \sqrt{p_3^2 + m_e^2 + 2eBn}$, and $\beta = T^{-1}$. For W's, we have $\Omega_W = \Omega_{sW} + \Omega_{0W}$

$$\Omega_{sW} = \frac{eB}{4\pi^2\beta}\int_{-\infty}^{\infty} dp_3 \ln[(1 - e^{-(\epsilon_{0q}-\mu_W)\beta})(1 - e^{-(\epsilon_{0q}+\mu_W)\beta})] + \frac{eB}{4\pi^2\beta}\sum_{n=0}^{\infty} b_n \int_{-\infty}^{\infty} dp_3 \ln[(1 - e^{-(\epsilon_q-\mu_W)\beta})(1 - e^{-(\epsilon_q+\mu_W)\beta})], \quad (2)$$

where again we sum over all Landau quantum numbers and the degeneracy factor is $b_n = 3 - \delta_{0n}$, with $\epsilon_{0q} = \sqrt{p_3^2 + M_W^2 - eB}$, and $\epsilon_q = \sqrt{p_3^2 + M_W^2 + 2eB(n + \frac{1}{2})}$.

The Euler-Heisenberg vacuum terms are, for the electron-positron field,

$$\Omega_{0e} = \frac{e^2B^2}{8\pi^2}\int_0^\infty e^{-m_e^2 x/eB}\left[\frac{\coth x}{x} - \frac{1}{x^2} - \frac{1}{3}\right]\frac{dx}{x}, \tag{3}$$

and for the charged gauge bosons,

$$\Omega_{0W} = -\frac{e^2B^2}{16\pi^2}\int_0^\infty e^{-M_W^2 x/eB} \times\left[\frac{1 + 2\cosh 2x}{\sinh x} - \frac{3}{x} - \frac{7x}{2}\right]\frac{dx}{x^2}, \tag{4}$$

which diverges at $B > B_c$, leading to a vacuum instability.

The mean density of particles minus antiparticles (average charge divided by e) is given by $N_{e,W} = -\partial\Omega_{e,W}/\partial\mu_{e,W}$. We assume that there is always a background charge of opposite sign, to preserve electrical neutrality. We have

$$N_e = \frac{eB}{4\pi^2}\sum_0^\infty a_n\left[\int_{-\infty}^\infty dp_3(n_e^+ - n_e^-)\right], \tag{5}$$

where $n_e^\pm = [\exp(E_q \mp \mu_e)\beta + 1]^{-1}$.

In the degenerate limit one gets $N_e = \frac{eB}{2\pi^2}\sum_0^{n_\mu} a_n\sqrt{\mu_e^2 - m^2 - 2eBn}$, where the integer $n_\mu = I[(\mu_e^2 - m^2)/2eB]$.

For W,

$$N_W = \frac{eB}{4\pi^2}\left[\int_{-\infty}^\infty dp_3(n_{0p}^+ - n_{0p}^-)\right] + \frac{eB}{4\pi^2}\sum_0^\infty b_n\left[\int_{-\infty}^\infty dp_3(n_p^+ - n_p^-)\right] \tag{6}$$

with $n_{0p}^\pm = [\exp(\epsilon_{0q} \mp \mu_W)\beta - 1]^{-1}$, $n_p^\pm = [\exp(\epsilon_q \mp \mu_W)\beta - 1]^{-1}$.

The magnetization is given by the contribution of electrons and charged vector bosons. It depends on the density of particles *plus* antiparticles, and it is $\mathcal{M}_{W,e} = -\partial\Omega_{W,e}/\partial B$, where (calling $\mathcal{M}_{0e,0W} = -\partial\Omega_{0e,0W}/\partial B$)

$$\mathcal{M}_e = -\frac{\Omega_{se}}{B} - \frac{e}{4\pi^2}\sum_0^\infty a_n\left[\int_{-\infty}^\infty dp_3\frac{eBn}{E_q}(n_e^+ + n_e^-)\right] + \mathcal{M}_{0e}, \tag{7}$$

and in the degenerate limit [12],

$$\mathcal{M}_e = \frac{e}{4\pi^2}\sum_0^{n_\mu} a_n\left(\mu_e\sqrt{\mu_e^2 - m^2 - 2eBn} - (m^2 + 4eBn)\ln\frac{\mu_e + \sqrt{\mu_e^2 - m^2 - 2eBn}}{\sqrt{m^2 + 2eBn}}\right) + \mathcal{M}_{0e}, \tag{8}$$

and

$$\mathcal{M}_W = -\frac{\Omega_W}{B} + \frac{e^2B}{8\pi^2}\left[\int_{-\infty}^\infty \frac{dp_3}{\epsilon_q^0}(n_{0p}^+ + n_{0p}^-)\right] - \frac{e^2B}{4\pi^2}\sum_0^\infty b_n\left(n + \frac{1}{2}\right)\left[\int_{-\infty}^\infty\frac{dp_3}{\epsilon_q}(n_p^+ + n_p^-)\right] + \mathcal{M}_{0W}. \tag{9}$$

It is now especially interesting to discuss the equation of state of the system. The total energy-momentum tensor, whose spatial diagonal components are the pressures along the coordinate axes, may be obtained by starting from the quantum statistical average $T_{\mu\nu} = \langle\mathcal{T}_{\mu\nu}\rangle_s$, where $\mathcal{T}_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial A^a_{\mu,\nu}}A^a_{\mu,\nu} - \delta_{\mu\nu}\mathcal{L}$ [13]. If $\mathcal{L}$ is the total Lagrangian, after doing the statistical average, its place in the energy-momentum tensor is taken by Ω (since $\Omega = -\beta^{-1}\ln\langle e^{\int_0^\beta dx_4\int d^3x\,\mathcal{L}(x_4,\mathbf{x})}\rangle_s$). In the present SU(2) $\times$ U(1) model, the only nonzero averaged components of the field tensor are those of the U(1) external magnetic field tensor $F_{\mu\rho}$ and then,

$$T_{\mu\nu} = (T\partial\Omega/\partial T + \mu_e\partial\Omega/\partial\mu_e + \mu_W\partial\Omega/\partial\mu_W)\delta_{4\mu}\delta_{\nu4} + 4F_{\mu\rho}F_{\nu\rho}\partial\Omega/\partial F^2 - \delta_{\mu\nu}\Omega. \tag{10}$$

For $F_{\mu\rho} = 0$, (10) reproduces the usual zero field case [13]. For the electrically charged particles, we obtain thus different equations of state for directions parallel and perpendicular to the magnetic field,

$$p_3 = -\Omega, \qquad p_\perp = -\Omega - B\mathcal{M}. \tag{11}$$

This anisotropy in the pressures $p_3, p_\perp$ leads to a magnetostriction effect in the quantum magnetized gas of charged particles. If (10) is taken as the Maxwell stress tensor (classical case), $\mathcal{M} < 0$ and $p_\perp > p_3$, which produces a flattening effect in white dwarfs and neutron star models [14,15]. In the present quantum case, for diamagnetic media also $\mathcal{M} < 0$ leading again to a flattening effect. But for positive magnetization, the transverse pressure exerted by the charged particles is smaller than the longitudinal one by the amount $B\mathcal{M}$. The extreme case is found for magnetic fields, $eB \gg T^2$, when the electrons are confined to the Landau ground state $n = 0$. (In what follows we will ignore the vacuum contribution to electron-positron pressure and magnetization, which is justified at the scale of densities and fields considered below.) We have $\Omega_e = -B\mathcal{M}_e$, where

$$\mathcal{M}_e = \frac{e}{2\pi^2}\left(\mu_e\sqrt{\mu_e^2 - m^2} - m^2\ln\frac{\mu_e + \sqrt{\mu_e^2 - m^2}}{m}\right) \tag{12}$$

and $\mu_e \simeq \sqrt{(2\pi^2 N_e/eB)^2 + m^2}$, N_e being the electron density. As $\mu_e^2 > m^2$, the expression (12) is always positive the system behaves as paramagnetic or ferromagnetic. But one of the most important effects we have in this limit is that the transverse pressure can vanish,

$$p_\perp = -\Omega_e - B\mathcal{M}_e = 0\,. \tag{13}$$

(This is the lower bound for the pressure. For fermions, the pressure cannot be negative.) The effect (13) is of pure quantum origin and it is easy to understand since all electrons are confined to the Landau ground state, and the quantum average of their transverse momentum vanishes. If we consider a white dwarf star in which the predominating contribution to the pressure is from the electron gas, the vanishing of $p_\perp$ means that the gravitational pressure (of order GM^2/R^4, where R is the geometric average radius of the star) cannot be compensated and an instability appears leading to a transverse collapse; i.e., the resulting object (a neutron star or a black hole) would be ellipsoidal, in this case stretched along the direction of the magnetic field, as a cigar. It is interesting to find the critical conditions for the occurrence of this confinement to the state $n = 0$, and in consequence, for the collapse. We have

$$n_\mu = I\left(\frac{\mu_e^2 - m^2}{2eB}\right) = \frac{2\pi^4 N_e^2}{e^3 B^3} \sim 4.75 \times 10^{-20}\frac{N_e^2}{B^3}\,, \tag{14}$$

and the condition $I(x) < 1$ might be satisfied in some astrophysical conditions. For example, for $N_e \sim 10^{30}$, $B = 3.36 \times 10^{13}$ G, it is enough that $B \gtrsim B_{c'}$ to satisfy it. For densities of the order of neutron stars, where a background of electrons and protons exist, if $N_e = 10^{39}$, the previous condition, if valid, would lead to $B > 10^{19}$ G.

The W population in the Landau ground state is significant if $d = \sqrt{M_W^2 - eB} \leq T$. In the degenerate limit, e.g., for $\sqrt{M_W^2 + eB}/T \gg 1$, one can neglect the contribution from excited Landau states and by taking only the $n = 0$ term in (9), one can approximate the first two terms, since the main contribution to the integrals comes from very small momenta,

$$\mathcal{M}_W = -\frac{eT}{4\pi}\sqrt{d^2 - \mu_W^2} + \frac{eBT}{4\pi}\frac{1}{\sqrt{d^2 - \mu_W^2}} + \mathcal{M}_{0W}\,. \tag{15}$$

The first term is the diamagnetic contribution which vanishes as $T \to 0$. The third is the vacuum contribution, which is asymptotically

$$\mathcal{M}_{0W} \sim -\frac{2\Omega_0}{B} - \frac{eM_W^2}{16\pi^2}\ln(M_W^2/eB - 1)\,,$$

whose most important term is the second one which contributes paramagnetically or ferromagnetically for $B > M^2/2e$, having a logarithmic divergence as $B \to B_c$. As the logarithm is negative for $B_c/2 < B \leq B_c$, that term has a negative contribution to the transverse pressure of vacuum for fields in that interval. The first term of $\mathcal{M}_{0W}$ contributes diamagnetically, but for $B \to B_c$ the dominant term in (15) is the second, which is also paramagnetic or ferromagnetic, having a stronger divergence (inverse square root) than the vacuum term. To have a more explicit form for (15), one must write μ_W in terms of the charge density. When confined to the Landau ground state the charge density of the system may be approximated as $N_W \sim eBT\mu_W/2\pi\sqrt{d^2 - \mu_W^2}$, from which $\mu_W^2 = d^2/[1 + \frac{e^2B^2T^2}{4\pi^2 N_W^2}]$, and

$$\mathcal{M}_W = -\frac{e^2T^2Bd}{4\pi\sqrt{4\pi^2N^2 + e^2B^2T^2}} + \frac{e^2BT}{4\pi d}\sqrt{1 + \frac{4\pi^2N^2}{e^2B^2T^2}} + \mathcal{M}_{0W}\,. \tag{16}$$

Taking $N \geq 10^{39}$, $T \sim 10^{-8}$ ergs, and $B \leq B_c$, one can neglect the unity in the square root and contributions from the first diamagnetic term in (15) and from $\mathcal{M}_0$, and one is left with

$$\mathcal{M}_W \simeq \frac{eN_W}{2d}\,. \tag{17}$$

The most important consequence is that the contribution of this magnetization to the transverse pressure of the W gas would be negative [see (9)], and if $\mathcal{M}_W B$ contributes more than the pressure of other species (the partial pressure $p_3 = \Omega_W$ even decreases as $B \to B_c$), an instability occurs since the total pressure would be negative. Thus, for stability (also to prevent W decay), we must assume some background able to keep the total pressure $p_\perp \geq 0$.

Some sort of Bose-Einstein condensation actually takes place [16] for bosons. For small momentum and magnetic fields strong enough $B \sim B_c$, the term $1/d$ dominates and the main contribution to the W propagator comes from the low momentum gauge bosons [12,16].

In the absence of a magnetic field, the quantum degeneracy of the W-boson sector leads to condensation, which at $T \simeq 0$ has been estimated [17] to occur induced by neutrino densities of order $n_{c\nu} = M_W^3/6\pi^2 \simeq 10^{45}\ \text{cm}^{-3}$. At any temperature, a spontaneous magnetization would appear in the condensate of charged bosons, say W^+, even at zero external field $H = B - 4\pi\mathcal{M} = 0$. This spontaneous magnetization could self-consistently maintain the microscopic field $B = 4\pi\mathcal{M}_{e,W}$.

Let us assume the magnetization large enough to maintain the internal field B self-consistently and very large densities, such that $\mu_e \gg m$. The dominant term in (9) is (17) $\sim eN_W/2d$. At $B \sim B_c$ G, we obtain that the self-consistent critical field is reached at an electron density $\sim 10^{48}$ electrons/cm^3.

At such field intensities $\mathcal{M}_W$ diverges, but if we write the self-consistency condition for the W sector, we have

$$B = 4\pi\mathcal{M} = 2\pi\frac{eN_W}{d}\,. \tag{18}$$

Let us write $eB = x^2 M_W^2$ and since $0 \leq x \leq 1$, we easily get

$$x^2 \sqrt{1 - x^2} = \frac{2\pi e^2 N_W}{M_W^3} = A\,. \tag{19}$$

Equation (19) has no real solution for $A > A_1 = 2\sqrt{3}/9$, corresponding to $N_W \sim 10^{48}$ cm^{-3}, and two solutions for $A < A_1$. For one solution, B increases with N_W up to $B = 2B_c/3$; for the other, B decreases as a function of N_W from the value B_c at $N_W = 0$. This indicates that the expression for the magnetization must include the contribution from Landau states other than the ground state, which leads to a diamagnetic response to the field. This would compensate the increase of the self-consistent field with increasing N_W to keep $B < B_c$.

This can be shown to occur from formula (9). If we call the ground state density N_{Wg} and the density in other Landau states N_{Wn} ($N_W = N_{Wg} + \sum N_{Wn}$), for $B > 2B_c/3$, $\partial B/\partial N_{Wg} < 0$, and $\partial B/\partial N_{Wn} > 0$ and excited Landau states start to be populated. The condensate in the ground state decreases in favor of the increase of the population in excited Landau states, which starts to grow and contribute diamagnetically to the total magnetization keeping $4\pi \mathcal{M} = B < B_c$. But for the system to react in this way, an enormous amount of energy (and angular momentum) would be required, of the order of, respectively, $N_W M_W$ and N_W (here we neglect the running of M_W). But the transverse collapse takes place at such densities: since the pressure comes essentially from the fermion (electron) background, the self-consistency condition $B = 4\pi \mathcal{M}_{e,W}$ leads to $p_3 = -\Omega \sim B\mathcal{M}_e$ and $p_\perp = p_3 - B(\mathcal{M}_e + \mathcal{M}_W) \lesssim 0$ and thus the system collapses.

Let us assume that in some stage of the early universe a very large external field $H \sim B_c$ was present. If $T \sim M_W$, as happened near and below the electroweak phase transition, using up the energy and angular momentum of the background radiation, $W^\pm$ pairs will be produced in the energetic more favorable Landau ground state (having a "mass" $= d$), and this process would be even more favored as the magnetic field approaches B_c even for lower temperatures. The magnetization $\mathcal{M}$ is given by an expression similar to the second term in (9), in which the expressions for the particle-antiparticle densities would be in equilibrium with the electromagnetic background radiation. This means that one must take the chemical potential as zero (equal number of $W^\pm$). Then, $\mathcal{M}_W \sim e^2 BT/4\pi^2 d$. The density of particle $+$ antiparticle pairs would then be $\simeq eBT \sim 10^{48}$ cm^{-3}, and the microscopic field $B < B_c$ starts to be maintained self-consistently. We would have the situation discussed in the previous paragraph. The process of W pair creation in the external field would lead to a creation of order from disorder, i.e., to an effective cooling of the subsystem considered, although due to similar reasons as before, $p_\perp \lesssim 0$ and the system becomes unstable and collapses.

We conclude, first, that if a degenerate electron gas is confined to its Landau ground state, its transverse pressure vanishes. This phenomenon establishes a limit to the magnetic fields expected to be observable in white dwarfs, and even in neutron stars. Second, the instability of the vacuum in magnetic fields $B \sim B_c$ in a hot and dense medium, is avoided, since the self-consistent magnetization prevents fields greater than $2B_c/3$, although under such conditions the system becomes also unstable and collapses.

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