Condensation of Microturbulence-Generated Shear Flows into Global Modes

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In full flux-surface computer studies of tokamak edge turbulence, a spectrum of shear flows is found to control the turbulence level and not just the conventional (0,0)-mode flows. Flux tube domains too small for the large poloidal scale lengths of the continuous spectrum tend to overestimate the flows and thus underestimate the transport. It is shown analytically and numerically that under certain conditions dominant (0,0)-mode flows independent of the domain size develop, essentially through an analog of Bose-Einstein condensation for the shear flows.

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The energy confinement of tokamaks is mainly controlled by small-scale ($\sim$cm) turbulence giving rise to the "anomalous transport." Analytical and computer-aided studies have found that the anomalous transport in turn is often controlled by fluctuating "zonal flows" [1,2], poloidal shear flows, which are assumed to have zero poloidal and toroidal mode numbers, but have radial scales similar to the turbulence. The present paper deals with the question of what happens to the shear flows when the turbulence scale lengths become very small compared to the plasma size such as in the tokamak edge, in particular, in future large machines. In this limit, the shear flows either can contain a finite $(0,0)$-mode component or may loose their global character and change into vortices with finite poloidal scale length, as will be demonstrated by numerical full flux-surface edge turbulence studies. Regarding their large poloidal and parallel but small (similar to the turbulence) radial scale lengths, these vortices should not be regarded as drift waves or convective cells [3] but rather as poloidally localized shear flows.

For cost reasons, the domains of turbulence simulations are usually thin flux tubes [4] or tokamak sectors [5], equivalent to flux tubes with special boundary conditions. The flux tube dimensions perpendicular to the magnetic field are of the order of $\sim$10 cm and they extend several $\sim$10 m along the magnetic field to accommodate the prevalent turbulent structures. For poloidally localized shear flows, however, these computational domains are not adequate and the flows always appear to extend across the complete flux tube. Since the $(0,0)$ mode is not damped as the other modes, it may therefore exert a strong stabilizing effect on the turbulence, which underestimates the transport compared to a true full flux-surface simulation. For the core turbulence, which has relatively large scales, full torus simulations [6,7] exhibit zonal flows extending over the complete flux surfaces. However, even for these scenarios it is not clear whether this remains true for a much larger ratio of flux-surface circumference to turbulence scale length or whether the flows have a finite scale length in the poloidal direction.

Following the numerical results, an analytic model for the shear flows is described, in which their poloidal and radial wave number spectra are controlled by the interplay of damping by the collisional electron response and ion dissipation, the linear response of the turbulence to the flows, and the excitation of flows by random fluctuations. Under certain conditions, a nonzero fraction of the flow energy is generated as $(0,0)$-mode flows, *regardless of the system size*. The mechanism is analogous to the Bose-Einstein condensation (BEC). The three effects acting on the flows take the role of absorption, stimulated, and spontaneous emission. For the BEC, a macroscopic fraction of the quanta is eventually scattered into the ground state because the state density *near* the ground state is too low to hold sufficiently many quanta under the prevalent conditions. For the flow system, the turbulence and shear flows form a feed back loop, which regulates the shear flow energy to the level needed for turbulence saturation. Condensation into the $(0,0)$ flow component occurs when a threshold in required flow energy is exceeded, and the $m \neq 0$ modes are unable to hold it.

Numerical results.—We discuss the results of turbulence simulations of the three dimensional electrostatic drift Braginskii equations with isothermal electrons (a subset of the equations of Ref. [8]) for two different cases: (a) the predominant instability is the resistive ballooning mode with the nondimensional parameters $\alpha_d = 0.2$, $\epsilon_n = 0.08$, $q = 5$, $\tau = 1$, $\eta_i = 1$, $\hat{s} = 1$; (b) there is a significant contribution from ion temperature gradient modes with $\alpha_d = 0.4$, $\eta_i = 3$, and the other parameters as in (a). The radial domain width in terms of the resistive ballooning scale length, L_0, in (a) was $24L_0$ and in (b) $48L_0$, the width L_θ perpendicular to $\mathbf{r}$ and $\mathbf{B}$ was $24L_0$ [only for (a)], $192L_0$, $384L_0$, and $768L_0$ [the corresponding tokamak minor radius is $a = L_\theta q/(2\pi)$]. For a definition of these parameters and units, see Refs. [8,9]. The parameters of the largest domain are consistent with the physical parameters $R = 3$ m, $a = 1.5$ m, $L_n = 12$ cm, $q_0 = 3.2$, $n = 3.5 \times 10^{19}$ m^{-3}, $Z_{\text{eff}} = 4$, $B_0 = 3.5$ T, and for (a) $T = 100$ eV, $L_0 = 5.1$ mm, $\rho_s = 0.58$ mm and for (b) $T = 200$ eV, $L_0 = 3.6$ mm, $\rho_s = 0.82$ mm. The perpendicular grid step size was $\Delta = 0.19L_0$ (a) and $\Delta = 0.38L_0$ (b). Parallel to the magnetic field 12 points per poloidal connection length were sufficient due to the large parallel scales of the ballooning modes. The largest runs had a grid of $128 \times 4096 \times 12$.

The dependence of the average $(0,0)$ shear flow energy density on the domain size, L_θ, is compared for the two cases in Fig. 1. In contrast to case (b), the shear flows in (a) are apparently not condensed into the $(0,0)$ mode since its energy density decreases proportional to $1/L_\theta \propto 1/a$, as is expected when a given shear flow energy density is distributed equally among an increasingly dense set of modes.

The k_θ spectrum of the flow velocity, $v = v_\theta = \partial_r \phi$, for the $L_\theta = 768 L_0$ runs [for case (a), see Fig. 2] exhibits a rise at low k_θ associated with the shear flows, different from the microturbulence fluctuations at $k_\theta \sim 1$. The mean square shear flow amplitude of the $m = 0$ mode in cases (a) and (b) is 0.3 and 0.9 times, respectively, the total mean square shear flow amplitude, suggesting strong condensation for (b). In both cases, the typical poloidal scale length of the $m \neq 0$ shear flows is roughly a factor of 10 greater than the scales of the turbulence. Failure of the computational domain to accommodate the scales of the uncondensed shear flows in case (a) results in an overestimate of the shear flow amplitude and hence in an underestimate of the anomalous transport. The particle flux for $L_\theta = L_r = 24 L_0$ was found to be 25% lower than for $L_\theta = 768 L_0$.

Analytic model.—As the first ingredient of a qualitative model for the poloidal shear flow spectra, we calculate the linear dispersion relation for finitely elongated shear flows. For clarity, in the linear electrostatic vorticity equation (with the plasma parameters absorbed into the units, see, e.g., [9]),

$$\nabla_\perp^2 (\partial_t + \gamma)\phi + \partial_\parallel^2 \phi = 0\,, \qquad (1)$$

we neglect temperature fluctuations, parallel ion velocity, drift effects, curvature, and magnetic fluctuations. These effects can lead to a real frequency (e.g., geodesic acoustic modes [10,11]) and to a coupling to parallel sound waves or Alfvén waves. The dissipative effects are the flow damping γ due to the ion dissipation assumed independent of the wave number and the damping due to the resistive electron response. As we will see below, for a potential condensation of the shear flows into global modes only a small region around a certain radial wave number, k_0, is important, which we set to one in (1) since its absolute value is not important. Because of the large poloidal wavelengths of the flows, we approximate $-\nabla_\perp^2 \approx k_0^2 = 1$ and obtain the dispersion relation

$$\omega_{\rm lin} = -i(\gamma + k_\parallel^2)\,, \qquad k_\parallel = \left(\frac{m}{q(r)} - n\right). \qquad (2)$$

The damping by the parallel resistive electron response is weak if either $m = n = 0$ holds or r is near a resonant surface defined by $m - nq(r_{mn}) = 0$. Focusing on a thin region around $r = r_0$ we obtain $k_\parallel \approx m\alpha_0(r - r_{mn})$, $\alpha_0 = -q'(r_0)/q(r_0)^2$. Hence the resistive flow damping is proportional to m^2, which is the reason for the poloidal elongation of the flows, i.e., their low mode numbers.

As reaction to a shear flow [1,2,12,13] the microturbulence may in turn influence the flows via the Reynolds stress [14,15] or the Stringer-Winsor mechanism due to poloidal pressure asymmetries [10,16]. Restricting ourselves to linear response theory, we assume a (coherent) flow amplification rate $g(k_r)$ depending only on the radial wave number k_r, because of the large poloidal correlation lengths of the shear flows. With the (incoherent) random forcing, f, representing the effect of the turbulence fluctuations, the equation for the flow amplitude in frequency space has the form of a Langevin equation,

$$\partial_t v = -i\omega_{\rm lin} v + g(k_r) v + f\,. \qquad (3)$$

From (3) we obtain a relation between the mean square spectra of the flows and the forcing in frequency space,

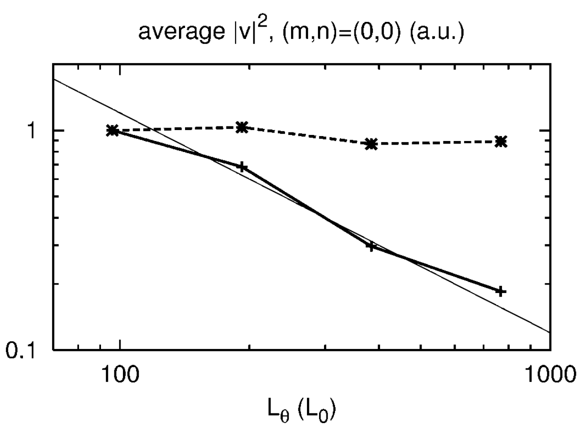

FIG. 1. $(0,0)$ shear flow energy density as a function of poloidal domain size L_θ for case (a) (solid line) without condensation and case (b) (dashed line) exhibiting condensation; the thin line is proportional to $1/L_\theta$.

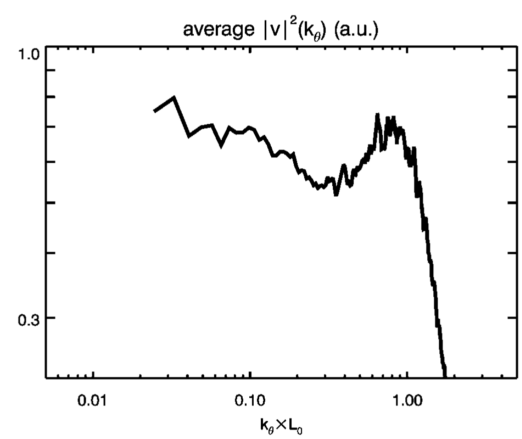

FIG. 2. Mean square shear flow amplitude as a function of k_θ for case (a) for $L_\theta = 768 L_0$. Note the greatly different scale lengths of the turbulence ($k_\theta > 0.3 L_0^{-1}$) and the shear flows ($k_\theta < 0.3 L_0^{-1}$).

$$\overline{|\hat{v}|^2} = \frac{\overline{|\hat{f}|^2}}{\omega^2 + [-i\omega_{\text{lin}} + g(k_r)]^2}. \quad (4)$$

Assuming that $\overline{|\hat{f}|^2}$ is independent of $\mathbf{k}, \omega$ (white noise), the integration of (4) over ω yields the relation between the mean square flow amplitude at an instant of time and the forcing,

$$\overline{|v|^2} = \frac{\overline{|\hat{f}|^2}\pi}{|-i\omega_{\text{lin}} + g(k_r)|}. \quad (5)$$

The flow intensity (5) replaces the Bose distribution in the BEC case. Both functions tend to infinity when the amplification (stimulated emission) terms approach the damping (absorption) terms. As long as every mode is net-damped at a rate independent of the system size, the energy density stored in $(0,0)$ modes must decrease proportional to the system size, since the total shear flow energy is distributed among an increasingly dense set of modes. However, analogous to the thermodynamic theory of the BEC, when the continuous flow spectrum is unable to hold the shear flow energy for the nonzero minimum net-damping rate and given random forcing, the nonlinear flow amplification term must adjust so that the remaining part of the flow energy is excited in the form of the most weakly damped modes, which are $(0,0)$ modes. Hence, to demonstrate the possibility of condensation, it has to be shown that the flow amplitude in $m \neq 0$ modes stays finite when the net-damping rate of the $m = 0$ modes tends to zero, in the limit of infinite system size or, equivalently, in the approximation of a continuous poloidal mode spectrum.

It is sufficiently general to assume that $g(k_r)$ has a maximum at $k_r = k_0$ of order of the turbulence wave numbers and is parabolic near that maximum, $g(k_r) = g_0 - g_1(k_r - k_0)^2$. The amplification terms will nearly cancel the damping terms only for wave numbers near k_0, which justifies the approximation $k_r \approx k_0$ which was made in the derivation of (2). For the following analysis we shift the k_r spectrum of the flows so that $k_0 = 0$. With $ik_r = \partial_r$, the operator in the denominator of (5),

$$(\gamma - g_0) - g_1\partial_r^2 + [m\alpha_0(r - r_{mn})]^2, \quad (6)$$

is the quantum mechanical Hamiltonian of the harmonic oscillator. Its eigenvalue for a mode with the "quantum numbers" (m, r_{mn}, l), $l \in \{0, 1, 2, \ldots\}$ is

$$\omega_l = \gamma - g_0 + 2\sqrt{g_1}\,|\alpha_0 m|\,(l + 1/2).$$

The sum over l of the eigenmode contributions to (5) at fixed (m, r_{mn}) results in a logarithmic divergence, which stems from the infinitely broad random forcing spectrum and infinitely fast turbulence response. Hence, we cut off ω_l at an appropriate ω_c depending on the turbulence. The sum is then approximated by an integral over l. The resulting amplitude associated with each pair (m, r_{mn}) is

$$\overline{|v|^2}(m, r_{mn}) = \frac{\overline{|\hat{f}|^2}\pi}{2\sqrt{g_1}\,|\alpha_0 m|} \ln\frac{\omega_c}{\omega_0} \quad (7)$$

with $\omega_0 = \gamma - g_0 + \sqrt{g_1}\,|\alpha_0 m| < \omega_c$. The density of rational surfaces is $|\alpha_0 m|$ for given m. Approximating the sum over all $\overline{|v|^2}(m, r_{mn})$ contributions to (5) with $m \neq 0$ by an integral (which becomes exact for infinite system size), the total instantaneous energy density of the flow modes with $m \neq 0$,

$$\overline{|v|^2}_{m\neq 0} = \int_{-m_c}^{m_c} |\alpha_0 m|\,\overline{|v|^2}(m, r_{mn})\,dm, \quad (8)$$

is obtained, where the integration interval is limited by the cutoff m_c defined by $\omega_0(m = m_c) = \omega_c$. With the minimum net-damping rate $\Omega = \omega_0(m = 0) = \gamma - g_0$ we obtain

$$\overline{|v|^2}_{m\neq 0} = \frac{\pi\overline{|\hat{f}|^2}}{2|\alpha_0|g_1}\left[\omega_c - \Omega\left(1 + \ln\frac{\omega_c}{\Omega}\right)\right]. \quad (9)$$

This expression converges to a finite value for $\Omega \to 0$. On the other hand, because the energy density of the $m = 0$ modes, which tends to infinity for $\Omega \to 0$ [the integral over k_r of (5) does not exist for $-i\omega_{\text{lin}} + g_0 = 0$], has to be finite, we always have $\Omega > 0$. If the turbulence saturation requires a higher flow level than (9) at $\Omega \to 0$, the description of the system by a continuum of poloidal mode numbers breaks down, the flow energy which can not be received by the $m \neq 0$ modes condenses into $m = 0$ modes, and simultaneously $\Omega \to 0$.

In a similar manner, it can be shown that in the limit of infinite system size the $m = 0$ condensate is *completely* contained in the $n = 0$ modes. Furthermore, there is no condensation of the radial wave numbers but the k_r spectrum becomes arbitarily narrow around the point of weakest net-damping for large system size.

Strictly speaking, condensation is unprovable by numerical studies, due to the restriction to finite system sizes. However, the validity of the individual parts of the model can be checked in the simulations. The localization of the $m \neq 0$ shear flows on resonant surfaces is obvious in a plot of the flow spectra versus radius (Fig. 3). The total flow amplitude associated with each (m, r_{mn}) quantum number in Eq. (7) (Fig. 4) has a much weaker slope than $k_\theta^{-2} \propto m^{-2}$ for $k_\theta, m \to 0$. Therefore the integral over all $m \neq 0$ shear flows (8) is expected to be finite in the limit of infinite system size [even if the estimate (7) for the individual amplitudes should be quantitatively wrong]. Furthermore, the integral is reasonably well approximated by the corresponding sum in the finite system. Consequently, the infinite system will have approximately the same ratio of $m = 0$ flow amplitude to $m \neq 0$ flow amplitude. Finally, we note that the numerical studies agree with the above analytical prediction, that the flow condensate exhibits a strong peaking in k_r for a sufficiently large system size.

Conclusions and consequences.—It has been shown numerically that in general the shear flows controlling the

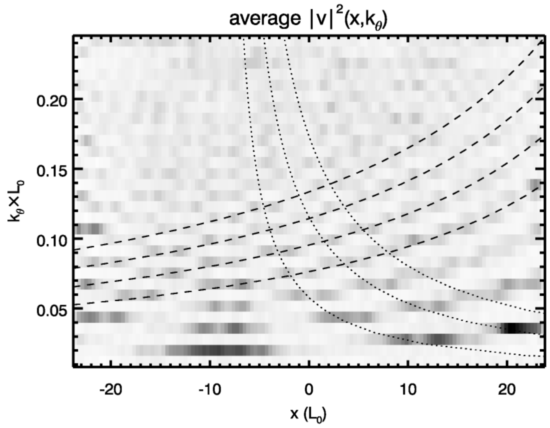

FIG. 3. Mean square shear flow amplitude as a function of k_θ and radius $x = r - r_0$ for case (b) for $L_\theta = 768L_0$. Note the localization of $k_\theta \neq 0$ flows on resonant surfaces. The resonances lie on the intersections of families of hyperbolas in the $x \times k_\theta$ plane, some examples of which are displayed.

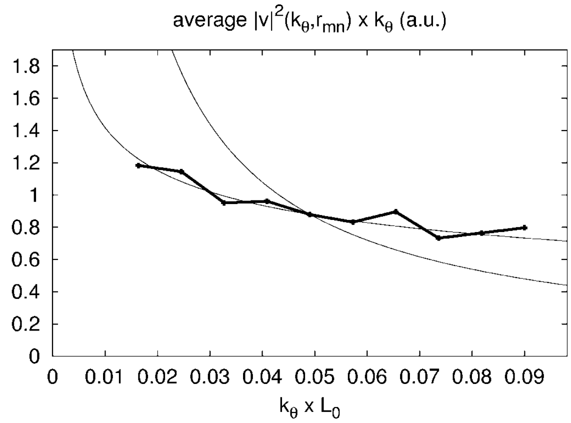

FIG. 4. Mean square shear flow amplitude on a single resonant surface multiplied with $k_\theta \propto m$ for case (b) for $L_\theta = 768L_0$. The thin lines are proportional to k_θ^{-1} (steeper curve) and $k_\theta^{-0.3}$ (flatter curve).

turbulence are not only $(0, 0)$ modes but rather consist of a spectrum of poloidal mode numbers. The $(m, n) \neq (0, 0)$ flows differ from drift waves or convective cells by their large poloidal [10 times larger than the turbulence (Fig. 2)] and parallel scale length, while their perpendicular scale length is similar to that of the turbulence. These shear flows are localized in the vicinity of resonant surfaces (Fig. 3). In the limit of large system size, a nonzero $(0, 0)$-mode amplitude develops only if the shear flows undergo a condensation into these modes, analogous to the Bose-Einstein condensation. Several features predicted by the analytic model have been reproduced by the numerical simulations.

Because of the canceling of damping and amplification terms, the $(0, 0)$ flow condensate is practically undamped. This means in quantum mechanical language that the rate of absorption and incoherent re-emission, the "collision" rate, vanishes. Hence, far ranging interactions or ordering effects might be mediated via the shear flow condensate (but not by the uncondensed flows that suffer collisions and are pinned to resonant surfaces). As a consequence in the simple system used here the k_r spectra become arbitrarily narrow.

Since the flows depend on the distribution of rational surfaces and mode numbers to accurately model the shear flow system in numerical studies, care has to be taken not to introduce spurious resonant surfaces or modes, e.g., by parallel extension of the flux tube [4,5]. Remarkably, it can be shown that increasing the flux tube length does not lead to the correct limit of large system sizes, since, e.g., for an infinitely long flux tube, condensation into $(0, 0)$ modes cannot occur.

Up to now, in flux tube based turbulence computations the shear flows were implicitly assumed to be global modes. With domain widths too small for the large poloidal scales of the continuous part of the flow spectrum, the flows *appear* to have zero poloidal and toroidal mode numbers. Such modes do not experience the resistive damping, which would reduce the flow amplitude in a full system. Hence, the simulations tend to overestimate the total flow amplitude, which may therefore exert a strong stabilizing effect on the turbulence. To avoid an underestimate of the transport, flux tube simulations have to be checked for influences of a finite poloidal scale length of the flows.

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