Spectrum of Coherent Structures in a Turbulent Environment

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The statistical properties of the turbulent field consisting of drift waves randomly interacting with a coherent structure are investigated. By using a nonperturbative method (analogous to the "semiclassical" approach in quantum mechanics), we calculate explicitly the generating functional of the correlations.

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Many experimental and numerical studies of fluid and plasma turbulence have revealed the presence of persistent coherent structures coexisting with the random phase fluctuations ([1,2]). In particular, it has been shown that the turbulent drift-wave potential in the outer region of the tokamak plasma can generate coherent structures [3]. It is generally admitted that perturbative methods (based on low orders in the series of cumulants) cannot describe coherent structures [1]. In this paper we present an analytical approach to the problem of determination of the statistical properties of the mixed state of "homogeneous" drift turbulence and coherent structures. The starting point is the observation that the coherent structure and the drift waves, although very different in form, are similar from a particular point of view: the former realizes the extremum of the action functional that describes the nonlinear evolution of the plasma and the latter obtain a value of the action very close to this extremum. Our approach is based on results from well-established theories: the functional statistical description of classical stochastic dynamical systems ([4], in the path integral formalism [5]; see also [6]); the perturbed inverse scattering transform method, allowing one to derive the perturbed form of nonlinear coherent structures [7]; the semiclassical approximation in the study of the quantum particle tunneling in multiple minima potentials [8]. In standard field theory terminology our method belongs to the "nonperturbative" approaches.

In previous works on the dilute gas of plasma solitons, the field has been represented as a sum of noninteracting solitons randomly distributed in the volume and with random velocities. A statistical analysis has been done for the nonlinear drift waves by Meiss and Horton [9]. Actually our work starts from the same nonlinear equation and potentially contains the analysis of [9], but we shall not consider this type of stochasticity and will focus on the manifestation of the drift-wave turbulence. Our primary objective is to determine the spectrum of a plasma vortex whose shape is randomly perturbed by interaction with random linear drift waves.

We consider the plasma confined in a strong magnetic field and the drift-wave electric potential in the transversal plane (x, y), where y corresponds to the poloidal direction and x to the radial one in a tokamak. The nonlinear drift equation (studied in Ref. [9]) is

$$(1-\rho_s^2\nabla_\perp^2)\frac{\partial\varphi}{\partial t}+v_d\frac{\partial\varphi}{\partial y}-v_d\varphi\frac{\partial\varphi}{\partial y}=0, \quad (1)$$

where $\rho_s = c_s/\Omega_i$, $c_s = (T_e/m_i)^{1/2}$, and the potential is scaled as $\varphi = \frac{L_n}{L_{T_e}}\frac{e\Phi}{T_e}$. Here L_n and L_{T_e} are, respectively, the gradient lengths of the density and temperature. The velocity is the diamagnetic velocity $v_d = \rho_s c_s/L_n$. The exact solution of the equation is $\varphi_s(y,t;y_0,u) = -3(\frac{u}{v_d}-1)\times \mathrm{sec}h^2[\frac{1}{2\rho_s}(1-\frac{v_d}{u})^{1/2}(y-y_0-ut)]$, where the velocity is restricted to the intervals $u > v_d$ or $u < 0$. The space "volume" on the y coordinate is $(-L/2, L/2)$ with $L \gg \rho_s$. We assume that u is very close to v_d, $u \gtrsim v_d$ (i.e., the solitons have small amplitudes). The nonlinear equations for the drift waves are known to admit as solutions irregular turbulent fields but also exact coherent structures of the type $\varphi_s(y,t;y_0,u)$, depending on the initial conditions. For a statistical ensemble of initial conditions both tendencies will be present, competing in the determination of the spectrum.

We now present an outline of the method. Instead of using directly the equation of evolution of the system [the original nonlinear equation for the field φ, (1)] we start by constructing the action functional of the system interacting with an external current. The dynamical equation (whose exact solution is the vortex soliton perturbed by interaction) is the Euler-Lagrange equation derived from the condition of extremum of this functional. By using the exponential of the action, we construct the generating functional of the irreducible correlations of φ. This functional contains all of the information on the coherent structure and the drift turbulence. The correlations are obtained via functional derivatives to the external current. The generating functional is by definition a functional integral over all possible configurations of the system and this integral must be calculated explicitly. The zeroth order of the stationary phase approximation (in function space) relies on the perturbed vortex soliton, i.e., on the solution which obtains the extremum of the action including the interaction with the external current. The drift waves (solutions of the linear part of the equation) do not exactly realize the extremum of the action functional, but obtain an action close to this extremum. This means that the drift waves are

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near the vortex soliton in the function space in the sense of the measure defined by the exponential of the action. It also suggests performing the functional integral with better approximation (instead of simply calculating the action along the soliton solution), which means to perform the integration over a function-space neighborhood of the vortex solution. This will automatically retain the drift waves in the generating functional of correlations which, thus, will contain information on both the coherent structure and the drift waves. To define the extension in the function space of the neighborhood around the soliton, we note that the measure (exponential of the action) has wild oscillations which suppress all configurations of the system which are far from the solution realizing the extremum (i.e., the vortex soliton).

To simplify the notations we rewrite Eq. (1) as $\hat{O}\varphi = 0$ where $\varphi(x,y,t)$ represents the "field" (coherent structure plus drift waves), and the operator $\hat{O}$ is the nonlinear operator of the equation. The statistical nature of our problem is taken into account in the construction of the action functional. The field φ obeys a purely deterministic equation, but the randomness of the initial conditions generates a statistical ensemble of realizations of the system evolutions (space-time configurations). We shall construct the action functional in the path-integral formalism ([5,10]). Every function from the statistical ensemble of realizations of the system's space-time configurations is discretized in space and time, so it will be represented as a collection of varables φ_i, each attached to the corresponding space-time point i. In this space of functions, the selection of the configurations which correspond to the physical ones (solutions of the equation of motion) is performed via the identification with Dirac δ functions, in every space-time point: $\prod_i \delta[\varphi_i - \varphi(x_i,y_i,t_i)]\delta[\hat{O}\varphi]$ and integration over all possible functions φ, i.e., over the ensemble of independent variables φ_i. Using the Fourier representation for every δ function, we get

$$\int \prod_i d\varphi_i \int \prod_i d\chi_i \, \exp[i\chi_i \hat{O}\varphi(x_i,y_i,t_i)] .$$

Going to the continuum limit, a new function appears, $\chi(x,y,t)$, similar to the Fourier variable conjugate of φ. The generating functional of the correlation functions is

$$Z = \int D[\varphi]D[\chi]\exp\left\{i \int d\mathbf{x}'dt' \, \chi(\mathbf{x}',t')\hat{O}\varphi(\mathbf{x}',t')\right\},$$

where functional measures have been introduced and $\mathbf{x} \equiv (x,y)$. The random initial conditions $\varphi_0(y)$ can be included by a Dirac δ functional: $\delta[\varphi(t_0,y) - \varphi_0(y)]$. This approach requires one to solve *exact* evolution equations starting from every realization in a statistical ensemble of initial conditions, followed by averaging. Instead of this exact treatment (accessible only numerically) we exploit the particularity of our approach, i.e., the connection between the functional integration and the delimitation of the statistical ensemble: the way we perform the functional integration is an implicit choice of the statistical ensemble. The inverse scattering transform in the presence of a perturbation obtains the generic modifications of the soliton: shape deformation (tail with a long plateau ending by an oscillation part), decrease of the amplitude, and radiation of linear waves. We choose to build implicitly the statistical ensemble, collecting all configurations which have the same type of deformations (given in our formulas by $\tilde{\chi}_J$). All of these configurations belong to the neighborhood of the extremum in function space. We take them into account, by performing the integration over this space. In doing so, we assume that the ensemble of perturbed configurations induced by an "external" excitation (J below) of the system is the same as the statistical ensemble of the system's configurations evolving from random initial conditions. The response of the system to an external excitation will be obtained by adding to the expression in the integrand at the exponential a linear combination representing the interaction of the fields φ and χ with external currents J_φ and J_χ. (More complex descriptions might be necessary for situations like the superstrong Langmuir turbulence, see [11].)

$$Z \to Z_J = \int D[\varphi(\mathbf{x},t)]D[\chi(\mathbf{x},t)] \exp\{iS_J\},$$
$$S_J \equiv \int d\mathbf{x}'dt' [\chi(\mathbf{x}',t')\hat{O}\varphi(\mathbf{x}',t') + J_\varphi\varphi + J_\chi\chi] . \tag{2}$$

For simplicity we shall consider in this work a single vortex soliton. To obtain the explicit form of Z_J, the perturbed soliton solutions φ_{Js} and χ_{Js}, depending on $\mathbf{J}$ (solutions of $\delta S_J/\delta\varphi = 0$, $\delta S_J/\delta\chi = 0$), must be introduced in the expression of the action S_J (which will be noted S_{Js}). After that we perform the expansion of the functions φ and χ about the coherent solution, $\varphi = \varphi_{Js} + \delta\varphi$ and $\chi = \chi_{Js} + \delta\chi$. Denoting $\delta\Phi$ the column matrix with elements $\delta\Phi_1 = \delta\varphi$, $\delta\Phi_2 = \delta\chi$, we have

$$Z_J = \exp(iS_{Js}) \int D[\delta\varphi]D[\delta\chi] \times \exp\left\{\int d\mathbf{x}'dt' \, \delta\Phi^T(\mathbf{x}',t')\left(\frac{\delta^2\hat{O}}{\delta\varphi\delta\chi}\bigg|_{\varphi_s,\chi_s}\right)\delta\Phi(\mathbf{x}',t')\right\}$$
$$= \exp(iS_{Js}) \frac{1}{2^n i^n} (2\pi)^{n/2}\left(\det\frac{\delta^2\hat{O}}{\delta\varphi\delta\chi}\right)^{-1/2}$$

since the integral is Gaussian [12]. The determinant is calculated using the eigenvalues of the operator $\frac{\delta^2\hat{O}}{\delta\varphi\delta\chi}\big|_{\varphi_s,\chi_s}$:

$$\det\left(\frac{\delta^2\hat{O}}{\delta\varphi\delta\chi}\bigg|_{\varphi_s,\chi_s}\right) = \prod_k \lambda_k . \tag{3}$$

In general, the direct (i.e., the vortex plus random drift waves) solution φ arises from an initial perturbation which, evolving in time, breaks into several distinct vortices (solitons) and drift waves, as shown by the inverse scattering method. The functionally conjugated ("regressive") function χ is, at $t = \infty$, a collection of vortices and drift-wave turbulence which, evolving backward in time toward $t = 0$, coalesce and build up into a single perturbation, the same as the initial condition of φ, up to the sign. In general χ has opposite topology to φ, which suggests $\chi = -\varphi$. This is confirmed by the homogeneous equation for χ:

$$(1 - \nabla_\perp^2)\frac{\partial \chi}{\partial t} + v_d \frac{\partial \chi}{\partial y} - v_d \varphi \frac{\partial \chi}{\partial y} = 0$$

which, when compared to the equation for φ, shows that $\chi = -\varphi$ is the solution. Since we are only interested in correlations of the physical field, φ we take $J_\chi = 0$ and note $J \equiv J_\varphi$. We have $\chi_{Js}(x, y, t) = -\varphi_s(x, y, t) + \tilde{\chi}_J(x, y, t)$, where $-\varphi_s(x, y, t)$ represents the "free" solution of the variational equation, i.e., the negative vortex (antisoliton), and $\tilde{\chi}_J(x, y, t)$ is the small modification induced by the inhomogeneous small term, $J(x, y, t)$. This part is of central importance since it will react to the derivation with respect to J, which gives the correlations, and in general is extremely difficult to calculate. We have adapted the results obtained by Karpman for the perturbed KdV soliton [7]. We note that $\varphi_{Js}(x, y, t) \equiv \varphi_s(x, y, t)$. Performing the calculation we obtain the following:

$$\frac{1}{2}\,\delta\Phi^T\left(\frac{\delta^2 S_J}{\delta\varphi\,\delta\chi}\bigg|_{\varphi_{Js},\chi_{Js}}\right)\delta\Phi = \frac{1}{2}\,\delta\Phi^T\begin{pmatrix}\hat{\gamma} & -\hat{\alpha} - \hat{\beta} \\ \hat{\alpha} - \hat{\beta} & 0\end{pmatrix}\delta\Phi\,,$$

where

$$\hat{\alpha} = (1 - \nabla_\perp^2)\frac{\partial}{\partial t} + v_d \frac{\partial}{\partial y} - v_d\left(\frac{\partial \varphi_{Js}}{\partial y}\right)$$
$$\hat{\beta} = \frac{1}{2} v_d\left(\frac{\partial \varphi_{Js}}{\partial y}\right), \qquad \hat{\gamma} = -2 v_d \chi_{Js}\frac{\partial}{\partial y}\,. \tag{4}$$

We change to the referential moving uniformly with the diamagnetic velocity, $t \rightarrow t$ and $y \rightarrow y - v_d t$, and neglect the slow motion of the soliton in the new frame. The equation for the eigenfunction w and eigenvalues λ can be rewritten as $w'' + (\lambda^2 t_1 + \lambda t_2 + t_3)w = 0$ with the notations

$$t_1(y) \equiv \frac{1}{v_d^2}\frac{h^2 - \varphi_s^2}{h^4} + \frac{2}{v_d}\frac{\varphi_s}{h^4}\tilde{\chi}_J, \qquad t_3(y) \equiv -\frac{3}{4}\frac{1}{h^2}\left(\frac{\partial \varphi_s}{\partial y}\right)^2,$$
$$t_2(y) \equiv -\frac{1}{v_d}\left(\frac{\partial \varphi_s}{\partial y}\right)\frac{2c - h}{h^3} + \frac{2}{v_d}\frac{(\frac{\partial \varphi_s}{\partial y})}{h^3}\tilde{\chi}_J - \frac{1}{v_d}\frac{1}{h^2}\left(\frac{\partial \tilde{\chi}_J}{\partial y}\right),$$

and $c \equiv \bar{k}_\perp^2$, $h = c + \varphi_s$. The function $U(\lambda; y) \equiv \lambda^2 t_1 + \lambda t_2 + t_3$ has singularities at the points where h vanishes, $h(\pm y_h) = 0$. The total space interval is then divided into three domains: $(-L/2, -y_h)$ (external left,"l"), $(-y_h, y_h)$ (internal,"i"), and $(y_h, L/2)$ (external right,"r"). Here "internal" and external refer to the region approximately occupied by the soliton.

The function w must vanish at the limits of the three domains and this condition generates discrete sets of eigenvalues. The infinite product of eigenvalues gives, for the external regions,

$$\prod_n \lambda_n^l \prod_{n'} \lambda_{n'}^r = \prod_n\left(\frac{2\pi n}{\alpha_1}\right)^2 \prod_n\left(1 - \frac{\sigma^2/(2\pi)^2}{n^2}\right) = \frac{\sin(\sigma/2)}{\sigma/2}\prod_n\left(\frac{2\pi n}{\alpha_1}\right)^2. \tag{5}$$

In the internal region, the infinite product of the eigenvalues is

$$\prod_n \lambda_n^i = \left[\frac{\sinh(\beta/2)}{\beta/2}\right]^{1/2}\prod_n\frac{(-i)2\pi n}{\alpha_c}\,, \tag{6}$$

where $\sigma = \int_{-L/2}^{-y_h} dy'\,t_2(y')[t_1(y')]^{-1/2}$ and $\beta = 2^{-1} \times \int_0^{y_h} dy'\,t_2(y')[-t_1(y')]^{-1/2}$. The functions α_1 and α_c will not contribute to the correlations. The generating functional becomes

$$Z_J = \exp(iS_{Js})\left(\prod_n\frac{(2\pi)}{i}\right)\left[\det\left(\frac{\delta^2 S_J}{\delta\varphi\,\delta\chi}\bigg|_{\varphi_{Js},\chi_{Js}}\right)\right]^{-1/2} = \text{const}\exp(iS_{Js})\left[\frac{\beta/2}{\sinh(\beta/2)}\right]^{1/4}\left[\frac{\sigma/2}{\sin(\sigma/2)}\right]^{1/2}, \tag{7}$$

where const $= \prod_n(\frac{(-i)\alpha_c}{2\pi n})^{1/2}\frac{\alpha_1}{n}$ will disappear after multiplying with Z_J^{-1}. The two-point correlation can be obtained by a double functional differentiation at the external current J:

$$\langle\varphi(y_2)\varphi(y_1)\rangle = Z_J^{-1}\frac{\delta^2 Z_J}{i\delta J(y_2)i\delta J(y_1)}\bigg|_{J=0}\,.$$

The double functional differentiation clearly emphasizes the mixing of the vortex solitons (terms from S_{Js}) with the turbulence formed by drift waves and wavelike soliton tails, i.e., configurations which are taken into account by the other two factors in Eq. (7). The expressions of $\tilde{\chi}_J$ and the formulas obtained by functional differentiation of the generating functional are very complicated and a numerical calculation is necessary. For Fig. 1 we have chosen a particular value of the soliton velocity (which also fixes its amplitude), $u = 1.9 v_d$, and let the variables y_1 and y_2 sample the one-dimensional volume of length $L = 0.1$ m. The physical parameters are chosen such that $\rho_s \approx 10^{-3}$ m and $v_d \approx 571$ m/s. Since we do not take

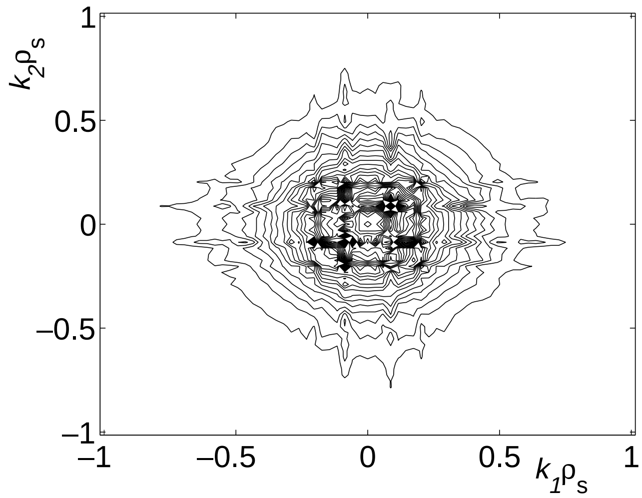

FIG. 1. Contour plot of the spectrum at $u = 1.9 v_d$.

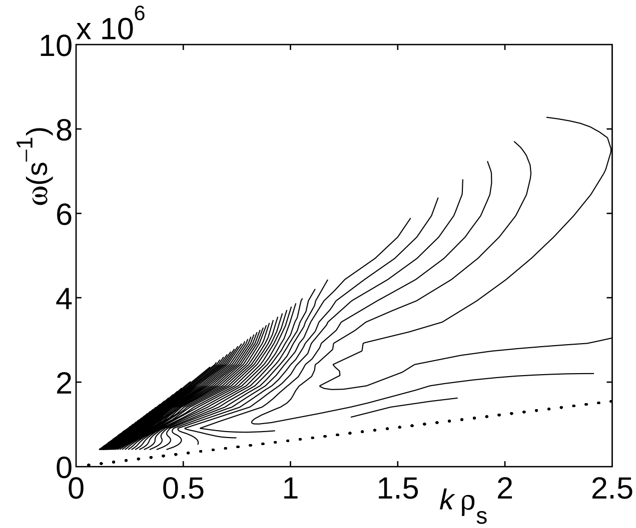

FIG. 2. Contour plot of the (k, ω) spectrum. The dotted line is $\omega = k v_d$.

the random position of the soliton center (and then do not average on it), the correlation is a function of two space variables (y_1, y_2). The contributions to the correlation from the last two factors in Eq. (7) have amplitudes similar or less by a factor of a few units, compared to the pure soliton. The factors coming from the internal part are peaked and localized on the soliton extension while the external part gives terms oscillating on (y_1, y_2). In wave-number space, there are contributions to both low-k and high-k regions. The spectrum of an unperturbed soliton is smooth and monotonously decreasing from the peak value at $\mathbf{k} = 0$. Figure 1 shows much more structure. In the low-k part there are many local peaks, an effective manifestation of the periodic character of the terms [as shown by Eq. (5)]. This arises from the discrete nature of the eigenvalues, which is induced by the second order differential operator and the vanishing of the eigenmodes at the positions of the singularities $\approx \pm y_h$. The singularities are generated by the vanishing of the norm of the operator $\hat{\alpha}$, which makes ambiguous the assumption of propagating wave character, $\partial_t = -v_d \partial_y$. The large-$k$ part mainly reflects the structure of the small-scale perturbation of the soliton's shape, comming from β-related terms. Figure 2 is a (k, ω) spectrum obtained from $\omega - ku = 0$ and repeating the calculations for various soliton velocities $u_{\max} > u > v_d$. Although we cannot afford high $u_{\max}$ since the expressions of $t_{1,2,3}(y)$ depend on the assumption $u \gtrsim v_d$, we note the local peaks in contrast to the "pure soliton" result of Ref. [9].

In conclusion, we have shown that a nonperturbative method ("semiclassical," in field theory) can be applied to plasma turbulence. This may considerably extend the studies beyond the perturbative renormalization.

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