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Method for Solving Moving Boundary Value Problems for Linear Evolution Equations

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We introduce a method of solving initial boundary value problems for linear evolution equations in a time-dependent domain, and we apply it to an equation with dispersion relation $\omega(k)$, in the domain $l(t) < x < \infty$, $0 < t < T$. We show that the solution of this problem admits an integral representation in the complex k plane, involving either an integral of $\exp[ikx - i\omega(k)t]\rho(k)$ along a *time-dependent contour,* or an integral of $\exp[ikx - i\omega(k)t]\rho(k,\bar{k})$ over a *fixed two-dimensional domain.* The functions $\rho(k)$ and $\rho(k,\bar{k})$ can be computed through the solution of a system of Volterra linear integral equations. This method can be generalized to nonlinear integrable partial differential equations.

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The aim of this Letter is to introduce a method for solving initial boundary value problems for linear evolution equations in a time-dependent domain. This method can be applied to an *arbitrary* linear evolution equation. For simplicity, we will consider the domain $l(t) < x < \infty$, $0 < t < T$, where $l(t)$ is a given differentiable function of t whose first derivative is monotonic, and T is a positive fixed constant. Although such problems appear naturally in applications, only the case of equations which are second order in the space derivative has been extensively studied. For problems with higher order derivatives, one encounters significant difficulties. For example, even if one wants to study such problems numerically, one must first resolve the question of identifying the number of boundary conditions needed at $x = l(t)$ for the problem to be well posed.

This method is the implementation in the case of moving boundary problems of the general approach for solving boundary value problems announced in [1]. Let $q(x_1, x_2)$ satisfy a linear partial differential equation (PDE) with constant coefficients in the domain $(x_1, x_2) \in \Omega$. The approach of [1] involves the following steps: (a) Given the PDE, construct two compatible eigenvalue equations. (b) Given Ω, perform the *simultaneous* spectral analysis of these two equations. Step (b) means constructing a function $\mu(x_1, x_2, k)$ which solves both eigenvalue equations and which for $(x_1, x_2) \in \Omega$, is bounded in k for all complex k, where k is the spectral parameter of the eigenvalue equations. It was shown in [2] that for polygonal domains the function μ is sectionally analytic, with jumps across fixed curves. Thus $\mu(x_1, x_2, k)$ can be constructed through the solution of a Riemann-Hilbert (RH) problem [3,4]. This, in turn, yields an integral representation for $q(x_1, x_2)$ which involves an integral along a fixed contour in the complex k plane.

The novelty of moving boundary value problems is that the above RH problem must now be replaced by either a RH problem formulated with respect to a *time-dependent* contour, or by a *d-bar problem* [3] formulated with respect to a *fixed two-dimensional domain.* This, in turn, yields an integral representation for $q(x_1, x_2)$ in the complex k plane which, if $l''(t) < 0$, involves an integral along a time-dependent contour, while if $l''(t) > 0$, involves an integral along the real k axis and a double integral over a fixed two-dimensional domain.

In order to minimize certain technical difficulties, we consider a general linear evolution equation of the *dispersive type*

$$\left[\partial_t + i\sum_{j=0}^{n}\alpha_j(-i\partial_x)^j\right]q(x,t) = 0\,, \qquad l(t) < x < \infty, \qquad 0 < x < T\,, \tag{1}$$

where all α_j's are real. We assume that $q(x,0) = q_0(x)$ is given and decaying for large x, and we look for a solution which decays for large x; we also assume that $l(0) = 0$.

The first step in the method introduced here is to show that if there exists a solution, this solution admits an integral representation with *explicit x and t dependence.* In order to describe this representation, we define certain contours and domains in the complex k plane, as well as the auxiliary function $S(k,\bar{k})$. Let $\omega(k) = \sum_{j=0}^{n} \alpha_j k^j$, and denote $k = k_R + ik_I$, $\omega(k) = \omega_R + i\omega_I$. The domains $D(t)$ and $E(t)$ are defined by $D(t) = \{k \in \mathbf{C}: \mathrm{Im}[\omega(k) - kl'(t)] > 0\}$; $E(t) = \{k \in \mathbf{C}: \mathrm{Im}[\omega(k) - kl'(t)] < 0\}$. $D_+(t)$ and $E_+(t)$ denote the parts of $D(t)$ and $E(t)$ in the upper half k plane, and similarly $D_-(t)$ and $E_-(t)$ denote the parts of $D(t)$ and $E(t)$ in the lower half k plane. $\partial D_+(t)$ is the oriented boundary of $D_+(t)$. L_1 is the oriented contour consisting of the part of the real axis which is also part of $D_+(T)$. L_2 is the oriented contour consisting of the part of the real axis which is also part of $D_+(0) \cap E_+(T)$. The orientation of the contour ∂D_+ is such that D_+ lies on the left-hand side of the increasing direction. The orientation of L_1 and L_2 is from the left to the right.

The function $S(k,\bar{k}) = S(\omega_I/k_I)$ is defined as the inverse of the function $\omega_I/k_I = l'(t)$, i.e.,

$$S(k,\bar{k}) = t \quad \text{iff } \omega_I/k_I = l'(t), \qquad 0 \le t \le T. \tag{2}$$

We note that ω_I/k_I is always well defined, while $S(\omega_I/k_I)$ is defined in $\mathbb{C}^+$ only for $k \in D_+(0) \cap E_+(T)$, and it takes values in $[0,T]$.

We will show that $q(x,t)$ admits the following representations:

(a) $l''(t) < 0$:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx - i\omega(k)t}\hat{q}_0(k)\,dk + \frac{1}{2\pi}\int_{\partial D_+(t)} e^{ikx - i\omega(k)t}\hat{Q}(k)\,dk\,, \tag{3}$$

where

$$\hat{q}_0(k) = \int_0^{\infty} e^{-ikx} q_0(x)\,dx\,,$$

$$\hat{Q}(k) = \sum_{j=1}^{n} \alpha_j[\hat{Q}_{j-1}(k) + k\hat{Q}_{j-2}(k) + \cdots + k^{j-1}\hat{Q}_0(k)] - l'(t)\hat{Q}_0(k)\,, \tag{4}$$

$$\hat{Q}_j(k) = \int_0^{T} e^{i\omega(k)t - ikl(t)}(-i\partial_x)^j q[l(t),t]\,dt\,, \quad j = 0,\ldots,n-1\,.$$

(b) $l''(t) > 0$:

$$q(x,t) = \frac{1}{2\pi}\Big\{\int_{-\infty}^{\infty} e^{ikx - i\omega(k)t}\hat{q}_0(k)\,dk + \int_{L_1} e^{ikx - i\omega(k)t}\hat{Q}(k)\,dk + \int_{L_2} e^{ikx - i\omega(k)t}\hat{Q}(k,S)\,dk + \int\int_{D_+(0)\cap E_+(T)} e^{ikx - i\omega(k)t}\frac{\partial\hat{Q}(k,S)}{\partial\bar{k}}\,dk \wedge d\bar{k}\Big\}, \tag{5}$$

where $dk \wedge d\bar{k} = -2i dk_R dk_I$, $\hat{Q}(k,S)$ is defined by an equation similar to the equation defining $\hat{Q}(k)$, but with T replaced by $S(\omega_I/k_I)$.

Equation (1) with $l(t) = 0$ is studied in [5], where it is shown that (a) $q(x,t)$ is given by Eq. (3) with $l(t) = 0$. (b) A necessary condition for well posedness is that N boundary conditions are given at $x = 0$, where $N = n/2$ if n is even, $N = (n+1)/2$ if n is odd and $\alpha_n > 0$, and $N = (n-1)/2$ if n is odd and $\alpha_n < 0$. (c) The functions $\hat{q}_0(k)$ and $\hat{Q}(k)$ satisfy a set of global relations. Using these relations, it is shown in [5] that the function $\hat{Q}(k)$ can be obtained from $\hat{q}_0(k)$ and the N given boundary conditions by solving a set of *algebraic equations.* As an example, suppose that $\partial_x^j q(0,t)$, $j = 0,\ldots,N-1$, are given. Then $\hat{Q}(k)$ involves the $n - N$ unknown functions $\partial_x^j q(0,t)$, $j = N,\ldots,n-1$. It is remarkable that the *single* global relation satisfied by $\hat{q}_0(k)$ and $\hat{Q}(k)$ is sufficient to determine *all* these unknown functions; see [5] for details.

We will show that the case $l(t) \neq 0$ is conceptually similar to the case $l(t) = 0$; in particular, the functions $\hat{q}_0(k)$, $\hat{Q}(k)$, and $\hat{Q}(k,S)$ satisfy certain global relations. Using these relations, it can be shown that a necessary condition for well posedness is that N boundary conditions are prescribed. However, in contrast with the case $l(t) = 0$, the functions $\hat{Q}(k)$ and $\hat{Q}(k,S)$ cannot be computed explicitly, but are determined through the solution of a system of *Volterra linear integral equations.*

These global relations are

(a) $l''(t) < 0$:

$$\hat{q}_0(k) = -\hat{Q}(k) + e^{i\omega(k)T}\int_{l(T)}^{\infty} e^{-ikx} q(x,T)\,dx\,, \quad k \in D_-(T)\,, \tag{6}$$

$$\hat{q}_0(k) = -\hat{Q}(k,S) + e^{i\omega(k)S}\int_{l(S)}^{\infty} e^{-ikx} q(x,T)\,dx\,, \quad k \in E_-(T) \cap D_-(0)\,. \tag{7}$$

(b) $l''(t) > 0$:

$$\hat{q}_0(k) = -\hat{Q}(k) + e^{i\omega(k)T}\int_{l(T)}^{\infty} e^{-ikx} q(x,T)\,dx\,, \quad k \in D_-(t)\,. \tag{8}$$

Using Eq. (8) it can be shown that, if $l''(t) > 0$, the spectral functions $\hat{q}_0(k)$ and $\hat{Q}(k)$ defined by Eqs. (4) satisfy the integral relations

$$\int_{\partial D_{-j}(t)} e^{ikl(t)-i\omega(k)t}[\hat{q}_0(k) + \hat{Q}(k)]\,dk = 0\,, \qquad j = 1,\ldots,n-N\,, \quad (9)$$

where $D_{-j}(t)$ are the simply connected components of $D_-(t)$.

For a given boundary value problem, these relations can be used to obtain a system of Volterra linear integral equations for the unknown part of the spectral function $\hat{Q}(k)$. Similarly if $l''(t) < 0$.

We now indicate how these results can be derived. Equation (1) admits the Lax pair

$$\mu_x - ik\mu = q, \qquad \mu_t + i\omega(k)\mu = -q_*\,, \quad (10)$$

where

$$q_*(x,t) = \sum_{j=1}^{n} \alpha_j[(-i\partial_x)^{j-1} + k(-i\partial_x)^{j-2} + \cdots + k^{j-1}]q(x,t)\,.$$

A solution of both Eqs. (10), bounded for all $k \in \mathbf{C}^-$, is given by

$$\mu_0(x,t,k) = -\int_x^\infty e^{ik(x-y)}q(y,t)\,dy\,. \quad (11)$$

A second solution of Eqs. (10) is

$$\mu(x,t,k) = e^{ik[x-l(t)]}F(t,k) + \int_{l(t)}^{x} e^{ik(x-y)}q(y,t)\,dy\,, \quad (12)$$

where $F(t,k) = \mu(l(t),t,k)$ satisfies the ODE

$$F_t + [i\omega(k) - ikl'(t)]F = l'(t)q(l(t),t) - q_*(l(t),t,k)\,. \quad (13)$$

In order to find solutions of Eq. (12) which are bounded for $k \in \mathbf{C}^+$, we must find solutions of Eq. (13) with this property. Such solutions always exist. If $l''(t) < 0$, these solutions are sectionally analytic, but if $l''(t) > 0$, there exists a domain of the complex k plane where these solutions are *not* analytic; this domain is $D_+(0) \cap E_+(T)$.

If $l''(t) < 0$, the sectionally analytic function $\mu(x,t,k)$ can be constructed from its *jumps* through the solution of a scalar RH problem. These jumps are of the form $\Delta = -e^{ikx-i\omega(k)t}\delta(k)$, where $\delta(k)$ is given by $\hat{q}_0(k)$, $k \in \mathbf{R}$, and $\hat{Q}(k)$, $k \in \partial D_+(t)$. The solution of this RH problem yields $\mu(x,t,k)$ and then the first of Eqs. (10) implies Eq. (3).

If $l''(t) > 0$, the sectionally bounded function $\mu(x,t,k)$ can be constructed from its jumps and from its d-bar derivative through the solution of a d-bar problem; this yields for $q(x,t)$ Eq. (5).

We note that since $\mu_0(l(t),t,k)$ satisfies Eq. (13), this function has a second representation. Comparing this second representation with Eq. (11) evaluated at $x = l(t)$ we find the global relations (6)–(8). Multiplying Eq. (8) by $e^{-i\omega(k)t+ikl(t)}$ we find, for $k \in \mathbb{C}^-$,

$$[\hat{q}_0(k) + \hat{Q}(k)]e^{-i\omega(k)t+ikl(t)} = e^{i\omega(k)(T-t)+ik(l(T)-l(t))} \times \int_{l(T)}^\infty e^{-ikx}q(x,T)\,dx\,. \quad (14)$$

We note that the terms $\hat{q}_0(k)$, $\hat{Q}(k)$, $\int_{l(T)}^\infty e^{-ikx}q(x,T)\,dx$ are well defined for $k \in \mathbb{C}^-$ and each of them has at least $O(1/k)$ decay as $k \to \infty$. Thus by Abel's theorem, each term appearing in (14) can be integrated as $k \to \infty$. Integrating along $\partial D_j(t)$, $j = 1,\ldots,n-N$, and noting that the last term in (14) is analytic and bounded in $D_j(t)$, we obtain Eq. (9). Similarly for $l''(t) < 0$.

Example 1: $q_t - q_{xxx} = 0$.—In this case, $n = 3$ and $\alpha_3 = 1$, thus $N = 2$. Since $\omega(k) = k^3$, the domain $D(t)$ is defined by $k_I[3k_R^2 - k_I^2 - l'(t)] > 0$; this domain, for the case $l'(t) < 0$, is shown in Fig. 1.

The function S is defined by

$$l'(t) = 3k_R^2 - k_I^2 \quad \text{iff } t = S(3k_R^2 - k_I^2)\,.$$

The domain $D_+(0) \cap E_+(T)$ for the case $l'(t) < 0$ is depicted in Fig. 2.

If $l'(t) < 0$, the contours L_1 and L_2 are empty. If $l'(t) > 0$,

$$L_1 = \left[\sqrt{\frac{l'(T)}{3}}, \infty\right] \cup \left[-\infty, -\sqrt{\frac{l'(T)}{3}}\right],$$

$$L_2 = \left[\sqrt{\frac{l'(0)}{3}}, \sqrt{\frac{l'(T)}{3}}\right] \cup \left[-\sqrt{\frac{l'(T)}{3}}, -\sqrt{\frac{l'(0)}{3}}\right].$$

The spectral function $\hat{Q}(k)$ is defined by

$$\hat{Q}(k) = \int_0^T e^{ik^3t-ikl(t)}\{[k^2 + l'(t)]q(l(t),t) - ikq_x(l(t),t) - q_{xx}(l(t),t)\}\,dt\,;$$

the spectral function $\hat{Q}(S,k)$ is defined by a similar expression, where T is replaced by $S(3k_R^2 - k_I^2)$.

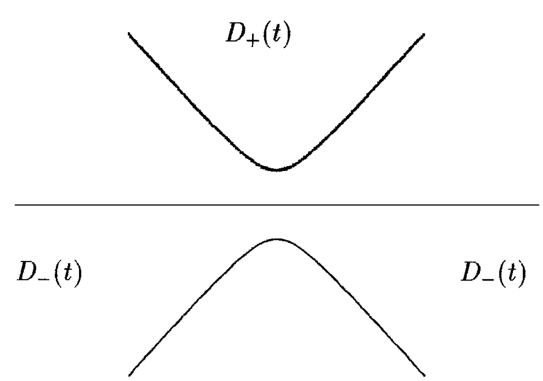

FIG. 1. Example 1: The domain $D(t)$ for $l'(t) < 0$. The curve is defined by $3k_R^2 - k_I^2 - l'(t) = 0$.

If $l'(t) < 0$, the representation of $q(x,t)$ is the following.

(a) $l''(t) < 0$:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ik^3t}\hat{q}_0(k)\,dk + \frac{1}{2\pi}\int_{\partial D_+(t)} e^{ikx-ik^3t}\hat{Q}(k)\,dk\,.$$

(b) $l''(t) > 0$:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ik^3t}\hat{q}_0(k)\,dk + \frac{1}{2\pi}\iint_{D_+(0)\cap E_+(T)} \times e^{ikx-ik^3t}\frac{\partial \hat{Q}(S,k)}{\partial \bar{k}}\,dk \wedge d\bar{k}\,.$$

In this case, Eq. (9) gives rise to one Volterra linear integral equation. As an illustrative example, suppose that $q_x(l(t),t) = f_1(t)$ and $q_{xx}(l(t),t) = f_2(t)$ are prescribed. Then Eq. (9) yields the following linear integral equation for the unknown $q(t) = q(l(t),t)$:

$$\frac{2\pi}{3}q(t) = F(t) - \int_0^t \left(\int_{\partial D_-(t)} e^{-ik^3(t-s)}\left\{[k^2 - l'(t)]e^{ik[l(t)-l(s)]} + \left[\frac{l'(t)}{3} - k^2\right]e^{ikl'(t)(t-s)}\right\}dk\right)q(s)\,ds\,, \qquad (15)$$

where $F(t)$ is the known function

$$F(t) = \int_{\partial D_-(t)} e^{-ik^3t+ikl(t)}\left\{\hat{q}_0(k) - \int_0^t e^{ik^3s-ikl(s)}[f_2(s) + ikf_1(s)]\,ds\right\}dk\,.$$

We conclude with some remarks. (1) Under the assumption of existence of a solution $q(x,t)$, Eq. (1) can be solved through a Fourier transform, yielding

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-i\omega(k)t}\hat{q}_0(k)\,dk - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-i\omega(k)t}\int_0^t e^{i\omega(k)s+ikl(s)}[q_*(l(s),s,k) - l'(s)q(l(s),s)]\,ds\,dk\,. \qquad (16)$$

We emphasize that this equation does *not* provide the spectral decomposition of $q(x,t)$. Furthermore, it is not clear from this formula how many boundary conditions are needed for a well-posed problem and how to determine the unknown functions $\partial_x^j q(l(t),t)$. (2) An important advantage of our method is that it can be applied to integrable nonlinear PDEs; see [1,6,7]. In this case, the scalar RH and d-bar problems must be replaced by their matrix-valued analogs. This implies that $q(x,t)$ does not admit an explicit integral representation but can be expressed in terms of the solution of a Fredholm linear integral equation. (3) For linear equations this method provides the constructive implementation and the generalization to concave domains of the celebrated Ehrenpreis principle [8]. (4) It is well known that the integral representation of the solution of a linear ordinary differential equation (ODE) in the complex plane provides a powerful tool for the study of many properties of this solution, including its asymptotic behavior. For linear equations this method provides the extension of such integral representations from ODE's to PDEs. We note that this extension takes the measure ρdk, where ρ is a constant, to either $\rho(k)dk$ or $\rho(k,\bar{k})dk_R dk_I$. (5) A RH problem is the fundamental object appearing in the solution of the initial value problem on the infinite line for integrable nonlinear equations in one space variable. Furthermore, RH problems have appeared recently in a variety of important applications in mathematical physics [4]. A d-bar problem is the fundamental object appearing in the solution of the initial value problem on the infinite plane for nonlinear integrable equation in two space variables [9,10]. The results presented here show that for the case of moving boundaries, d-bar problems play a crucial role even for linear PDEs in one space variable. (6) An important advantage of the representations (3) and (5) is that they have *explicit* x and t dependence. Similarly, the matrix RH and d-bar problems associated with the analogous problem for the nonlinear integrable equations have an explicit x and t dependence. This implies that it is possible to study the long-time asymptotic behavior of the solution. For linear equations, this can be achieved using the steepest descent method; for the RH case for nonlinear equations, it can be achieved using the elegant nonlinearization of the steepest descent method of Deift and Zhou [11]. (7) The physical difference between the two cases $l''(t) < 0$ and $l''(t) > 0$ can be understood by considering a piston positioned at $x = l(t)$. This piston excites waves whose

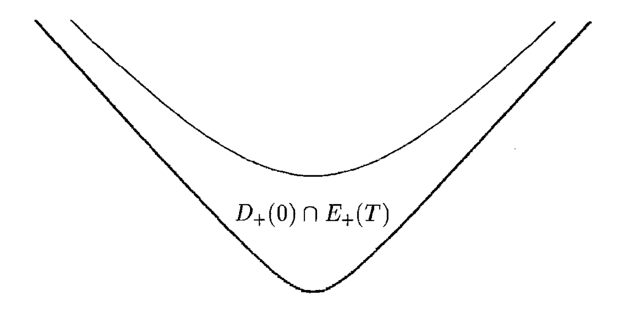

FIG. 2. Example 1: The domain $D_+(0) \cap E_+(T)$ for $l'(t) < 0$, bounded by the curves $3k_R^2 - k_I^2 - l'(0) = 0$ and $3k_R^2 - k_I^2 - l'(T) = 0$. The dotted line is R.

velocity is $l'(t)$. The group velocity of these waves is $\omega'(k)$, thus if $l''(t) > 0$, the associated rays intersect again with the piston.

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