Simple Model of Intermittent Passive Scalar Turbulence

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The passive scalar convection by chaotic two-dimensional incompressible flow is studied. Analytically solvable equations are suggested to describe the evolution of the probability density functions of tracer gradients and power spectra. The parameters of the model are expressed explicitly via the correlation functions of the velocity field. The multifractal spectrum $f(\alpha)$ of the scalar dissipation field is calculated; strict multifractality holds only for small values of α. Stationary and exponentially decaying power spectra of the scalar are obtained.

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The problem of the turbulent advection of a passive scalar ϕ (e.g., a dye concentration, temperature, etc.) has a long history and has been studied intensively due to its great practical importance. During the past few decades, the efforts have been concentrated on the analysis of the intermittent structure of the tracer distribution [1–9]. Perhaps the most immediate evidence of the intermittency is the multifractal structure of dissipation fields [1–4]; the scalar dissipation is defined as $\kappa|\nabla\phi|^2$, where κ is the molecular diffusivity. The Batchelor $1/k$ law for stationary tracer power spectra [5] and its exponential tail have been also frequently disputed subjects [6–8]. The same applies to the exponential decay of scalar fluctuations in the absence of an external source of dye [7,9]. Theoretical studies have been mainly based on the Kraichnan model [6] assuming velocity field to be delta correlated in time. This model has led to several important analytic results; however, the rigorous approach has still been unable to relate the parameters characterizing the above-mentioned phenomena directly to the correlation functions of the velocity field. The main tool for the multifractal analysis has been the generalized Baker map model, which has been also used to calculate the probability density function (PDF) of largest Lyapunov exponents [2,3].

In this Letter we suggest a simple equation to describe the PDF of tracer gradients, which is further used to calculate the multifractal spectrum of the scalar dissipation field. A similar equation is derived for the scalar power spectrum and used to study the long-time decay of the scalar; the stationary power spectrum in the presence of a stationary tracer source is also obtained. The parameters of the model are expressed directly via the correlation functions of the velocity field. The results are in good agreement with the earlier experimental, analytic, and numerical results. The method can be extended to cover more realistic physical conditions. In addition to the straightforward applications (heat and pollutant transfer in ocean and atmosphere, chemical mixing, etc.), it can be used to approach the more complex problems of hydrodynamic turbulence and magnetic dynamos.

The equation of our interest is

$$\frac{\partial\phi}{\partial t} + \boldsymbol{v}\cdot\nabla\phi = \kappa\nabla^2\phi + g, \tag{1}$$

where g is a source of passive scalar. We consider a two-dimensional (2D) problem with a chaotic isotropic single-scale incompressible velocity field $\boldsymbol{v}(\boldsymbol{r},t)$ in the absence of Kolmogorov-Arnol'd-Moser surfaces. Thus, we assume that the Fourier spectrum of the velocity field is constrained into one octave of wavelengths of unit length. These assumptions are mostly for the sake of simplicity; the approach can also be extended to the three-dimensional (3D) geometry and to the velocity fields with power spectra and intermittent structure. We consider the large Peclet number limit, $\kappa \ll \langle|\boldsymbol{v}|\rangle$, when the problem of finding the PDF of passive scalar gradients and power spectra can be reduced to the problem of finding the PDF of stretching factors of fluid elements $\rho(r,t) = \exp(h_m t)\cos\varphi$. Here, h_m stands for the largest Lyapunov exponent, and φ is the angle between the respective eigenvector and a fluid element. We consider the stretching factors and not the largest Lyapunov exponents, because, in the absence of molecular diffusivity, they are directly equivalent to the passive scalar gradients. Besides, they are easily tractable via the study of a fluid line evolution; this idea has been used to calculate the Kolmogorov entropy in the 2D quasistationary flow [10].

First, we study the case when there is no source of dye, and at the initial moment $t = 0$ there was a uniform gradient of dye concentration,

$$g \equiv 0, \qquad \nabla\phi|_{t=0} = \boldsymbol{e}_x, \tag{2}$$

where $\boldsymbol{e}_x$ is the unit vector along the x axis. We assume that dye has been convected long enough to create small-scale structures, but not too long, so that the smallest created scales are still longer than the dissipation scale $\sqrt{\kappa t}$. We obtain the PDF of tracer gradients and study the multifractal structure of the passive scalar dissipation. Further, we consider the long-time decay of the tracer fluctuations $\langle\psi^2\rangle$ (assuming $\langle\psi\rangle = 0$), under the same initial conditions. Finally, we analyze the steady-state power spectrum of the passive scalar field in the case of a statistically stationary single-scale source of dye,

$$g = g(\boldsymbol{r}, t). \tag{3}$$

The approach is based on a simple diffusion-convection equation, which we suggest to describe the distribution of fluid elements over the stretching factor ρ. We derive it both for the Kraichnan model and for velocity fields of finite correlation time. First, we consider the case of real velocity fields of finite correlation time τ, which is used as the unit time, i.e., $\tau = 1$.

The relationship between the stretching of fluid elements and passive scalar gradients has been exploited in many papers [4,7,11–14]. Following the idea of the previous studies, we notice that neglecting the molecular diffusivity, at a fluid particle, the modulus of the dye gradient evolves in the same way as the length of an infinitesimal fluid element $\delta \boldsymbol{r}(t)$, initially perpendicular to the gradient $[\delta \boldsymbol{r}(0) \perp \nabla \phi(0)]$: $|\nabla \phi| \propto |\delta \boldsymbol{r}(t)|$. Indeed, the fluid parallelogram defined by the initially perpendicular vectors $\delta \boldsymbol{r}(t)$ and $\delta \boldsymbol{r}_\perp(t)$ preserves its area $\delta S = |\delta \boldsymbol{r}(t)| \cdot |\delta \boldsymbol{r}_\perp(t)| \sin\alpha$, where α is the angle between the vectors. On the other hand, at the fluid particle, the dye concentration remains unchanged and $|\delta \boldsymbol{r}_\perp(t)| \sin\alpha \propto |\nabla \phi|^{-1}$. Thus, with the proper choice of units and neglecting the molecular diffusivity, the stretching factors $\rho = |\delta \boldsymbol{r}(\boldsymbol{r}_0, t)|/|\delta \boldsymbol{r}(\boldsymbol{r}_0, 0)|$ and dye gradients are equivalent to each other. Although the formal equivalence holds for the fluid elements, initially parallel to the isolines of $\phi(\boldsymbol{r}, 0)$, statistically the orientation given by isolines has no preference over the other directions.

Let us define $l(\rho, t) d\rho$ as the average total length of those pieces of a fluid line, for which $\rho \in [\rho, \rho + d\rho]$ (the length is reduced to the initial length of the fluid line L_0). Then, the PDF of stretching factors and dye gradients is given by $\rho^{-1} l(\rho)$. We consider time increments $\Delta t = 1$, and study the change of the state of fluid elements; we neglect the time correlation on time scales longer than $\tau = 1$. Let $p(q)dq$ denote the probability of stretching a fluid element by a factor of q. Then we can write $l(\rho, t + 1) d\rho = \int p(q) l(\rho/q, t)\, d(\rho/q) q\, dq$, or, introducing $\sigma = \ln\rho$ and $\lambda(\sigma, t) = l(\exp\sigma, t)$,

$$\lambda(\sigma, t + 1) = \int \lambda(\sigma - \ln q, t) p(q)\, dq\,. \tag{4}$$

The initial condition (2) can be rewritten as

$$\lambda(\sigma, 0) = \delta(\sigma)\,. \tag{5}$$

This system can be solved via Fourier transform $\lambda(f, t) = \int \lambda(\sigma, t) \exp(-if\sigma)\, d\sigma$, leading to

$$\lambda(f, t) = \Pi(f)^t,$$
$$\Pi(f) = \int p(\exp q) \exp(q - ifq)\, dq\,. \tag{6}$$

Upon taking inverse Fourier transform, denoting $h = \sigma/t$, and applying the saddle-point method on the long-time limit, expression (6) yields

$$\lambda \approx (\partial^2 F / t \partial f^2)^{-1/2} \exp\{F[f_0(h), h] t\},$$

where $F(f, h) = \ln[\int_{-\infty}^{+\infty} p(\exp q) \exp(q + fq)\, dq] - fh$, and $f_0(h)$ is the solution to the equation $\partial F/\partial f = 0$. This result has the same form as that obtained for the PDF of largest Lyapunov exponents $p(h_m, t)$ via the generalized Baker map model [3]. It should be noted that the functions $\lambda(h)$ and $p(h_m, t)$ are in fact distinct, even asymptotically at $t \to \infty$. This is caused by the presence of fluid elements, almost perpendicular to the eigenvector of the largest Lyapunov exponent. The function $F[f_0(h), h]$ has an additional advantage of being related directly to the correlation properties of the velocity field. For a typical localized correlation function $p(q)$, function $F(f, h)$ grows linearly at $f \to \pm\infty$. Thus, $f_0(h)$ is defined only for a finite range of the values of h, and outside of that region, $\lambda \equiv 0$.

In what follows, we consider a simplified version of Eq. (6), when $\Pi(f) = -iuf - Df^2$, where u and D are constants, and discrete increments are replaced by a time derivative. Then, Eq. (4) can be rewritten as

$$\frac{\partial \lambda}{\partial t} + u \frac{\partial \lambda}{\partial \sigma} = D \frac{\partial^2 \lambda}{\partial \sigma^2}\,. \tag{7}$$

Such a simplification will arise in two cases: (i) on long-time limit, when the function $\lambda(\sigma, t)$ has a smoothed profile and we can neglect the higher terms in the expansion of $\Pi(f)$ resulting in

$$u = \int p(k) \ln k\, dk\,,$$
$$D = \frac{1}{2} \int p(k) (\ln k)^2\, dk\,, \tag{8}$$

(ii) for the Kraichnan model, when $\Pi \equiv -iuf - Df^2$, with

$$u\delta(t - t') = \tfrac{1}{4} \langle [v_{r\tau}(t) + v_{\tau r}(t)][v_{r\tau}(t') + v_{\tau r}(t')] \rangle\,,$$
$$D\delta(t - t') = \tfrac{1}{2} \langle v_{rr}(t) v_{rr}(t') \rangle\,, \tag{9}$$

where indices r and τ denote the components of the tensor $\nabla \boldsymbol{v}$. This result can be obtained via the Fourier transform of the expression $\lambda(\sigma, t) = \langle \delta[\sigma - \ln|\delta \boldsymbol{r}(t)|] \rangle$, where $\delta \boldsymbol{r}(t) = \delta \boldsymbol{r}(0) \exp(\int_0^t \nabla \boldsymbol{v}\, dt')$, at the limit $t \to 0$. Here, the correlation time is zero, and, hence, time is measured in arbitrary units.

As discussed earlier, Eq. (7) describes the PDF of dye gradient for $\kappa = 0$. In order to handle the case of a nonzero molecular diffusivity, we apply Eq. (7) to the evolution of the quantity $\Phi = E_k k$, where E_k is the tracer power spectrum and k the modulus of the wave vector. While the legitimacy of such an approach can be questioned for $k \approx 1$, it is motivated for $k \gg 1$. Indeed, following the basic idea of Batchelor [5], for small wavelengths, the sinusoidal patterns are stretched by fluid almost homogeneously, and the wave vector is changed by the same factor as the local dye gradient. Meanwhile, the amplitude of tracer density oscillations (and, hence, the quantity $E_k k$) remains unchanged. The dissipation due to the molecular diffusivity leads to the exponential decay of sinusoidal fluctuations at the rate equal to $k^2\kappa$. The respective decay rate of Φ is $2k^2\kappa$. Upon substituting $k = \exp(\tilde{\sigma})$, we obtain

$$\frac{\partial\Phi}{\partial t} + u\frac{\partial\Phi}{\partial\tilde{\sigma}} = D\frac{\partial^2\Phi}{\partial\tilde{\sigma}^2} - 2\kappa\exp(2\tilde{\sigma})\Phi\,. \tag{10}$$

This equation is our main tool to analyze the power spectra.

The remainder of the Letter is devoted to the analysis of the consequences of Eqs. (7) and (10). First, we proceed to the calculation of the multifractal spectrum $f(\alpha)$ of the tracer dissipation field. Here we treat $\lambda(\sigma,t)$ as the PDF of dye gradients, $\sigma = \ln|\nabla\phi|$, and consider the initial conditions (5). In that case, Eq. (7) can be directly solved:

$$\lambda = (\pi Dt)^{-1/2}\exp\left[-\frac{(\sigma - ut)^2}{Dt}\right]. \tag{11}$$

Further we use the obtained PDF to derive the multifractal spectrum. To this end, we make use of the pattern formed by fluid curves, which were originally straight lines, separated by unit length, and perpendicular to the gradient of the dye concentration. We shall study a cross section of the dissipation field, and the dependence of the local value of σ on the coordinate ξ along the cross section. First, we note that the characteristic fluctuation amplitude of the dye concentration is 1. Indeed, when the fluid lines evolve, they will be folded; typically, the density variations of the order of unity are embraced between two approaching each other pieces of the curve. Thus, on the cross section, the characteristic scale of dye density variations is $\delta \approx 1/k \approx \exp(-\sigma)$. The small-scale variations of the function $\sigma(\xi)$ are described by the same scale. However, the function $\sigma(\xi)$ exhibits long-range correlations, as well, because two close to each other pieces of a fluid curve are stretched in a similar way. It can be argued that, in rescaled coordinates $\zeta = \int^{\xi}\xi'\exp[\sigma(\xi')]\,d\xi'$, function $\sigma(\zeta)$ is a random Brownian function.

The overall scalar dissipation in a region $[\xi,\xi + r]$, $r \lesssim 1$, can be assessed as $W_r(\xi) = \int_{\xi}^{\xi+r}\kappa k^2\,d\xi \approx \kappa k_{0r}^2\delta_0 = \kappa k_{0r}$, where k_{0r} is the maximal value of k over the given region, and $\delta_0 = 1/k_{0r}$. In order to determine the multifractal spectrum $f(\alpha)$, we need to know the probability,

$$p(r,\alpha) \propto r^{1-f(\alpha)}, \tag{12}$$

that the normalized dissipation $w_r(\xi) = W_r(\xi)/W_1(\xi)$ scales as αth power of r, i.e., $w_r \in [r^\alpha, 2r^\alpha]$. Because of the approximate equality $W_r(\xi)/W_1(\xi) \approx k_{0r}/k_{01}$, this probability can be calculated as

$$p(r,\alpha) = \begin{cases} L(\rho_0)r, & L(\rho_0)r \ll 1 \\ \exp[-L(\rho_0)r], & L(\rho_0)r \gg 1 \end{cases}$$
$$\text{with } \rho_0 = k_{01}r^\alpha. \tag{13}$$

Here $L(\rho_0) = \int_{\rho_0}^{\infty} l(\rho)\,d\rho$ is the overall length per unit area of those parts of the fluid curves, which are stretched more than a prefixed factor ρ_0. Indeed, $L(\rho_0)$ is the estimate for the number, how many times a cross section of unit length is intersected by the fluid curves of $\rho > \rho_0$. If the average number $N = Lr$ of such intersections per region of size r is very large ($N \gg 1$), the probability that there is only one intersection per region is exponentially small, $p \approx \exp(-N)$. At the opposite limit of $N \ll 1$, it can be assessed simply as N; at the marginally applicable limit of both expressions $N \approx 1$, we have a rough estimate of $p \approx 1$. According to (11),

$$L = \frac{1}{2}\exp\left[\left(u + \frac{D}{4}\right)t\right] \times \left[1 - \operatorname{erf}\frac{\sigma_0 + (u + D/2)t}{\sqrt{Dt}}\right], \tag{14}$$

where $\sigma_0 = \ln\rho_0$. At large values of σ_0, the asymptotics of Eq. (14) is given by $L \approx \exp[-(\sigma_0 + ut)^2/Dt - \sigma_0][1 + 2(\sigma_0 + ut)/Dt]^{-1}$; substituting $\sigma_0 = -\alpha|\ln r| + \ln k_{01}$ we obtain

$$p(r,\alpha) \approx r^{1-\alpha(\sqrt{1+4u/D}-\alpha|\ln r|/Dt)}. \tag{15}$$

Here we have also substituted the value of k_{01}, which has been calculated by noting that $p(1,\alpha) \approx 1$.

In its strict sense, multifractality assumes that $p(r,\alpha)$ is a power law of r; according to Eq. (15), this is valid for small values of α, $\alpha \ll Dt\sqrt{1 + 4u/D}/|\ln(r_0)|$, where r_0 is the smallest considered space scale. Under this assumption, expressions (12) and (15) yield

$$f(\alpha) = \alpha\sqrt{1 + 4u/D}\,; \tag{16}$$

remember that this expression assumes $Lr \ll 1$, and, hence, $f(\alpha) \lesssim 1$. On the other hand, slight deviations from multifractality in its strict sense may remain unnoticed when performing numerical schemes of obtaining multifractal spectra. Therefore, it makes sense to calculate the "average" value of $f(\alpha)$, which might be obtained in experiments:

$$\langle f(\alpha)\rangle = \ln[p(r_0,\alpha)/r_0]/|\ln r_0|\,. \tag{17}$$

In Fig. 1, the resulting curve is compared with the experimental results [1]. In both cases, there is a nearly linear part $f \propto \alpha$ (which is the only part of the curve corresponding to a strict multifractality), and there is a rapid [according to Eq. (13) exponential] falloff at large values of α. The differences can be attributed to (i) the fact that expression (13) is valid only at the asymptotic limit $r_0 \to 0$ (the effect of finite inertial range has been modeled by shifting the curves up and right), and (ii) the different dimensionalities of the velocity field.

Further we consider the long-time decay of the dye fluctuations. As discussed earlier, this problem is modeled by Eq. (10). The initial condition is given by Eq. (5): $\Phi(\tilde{\sigma},0) = \delta(\tilde{\sigma})$. The solution can be found via Laplace transform $\Phi_\omega = \int_0^\infty \Phi\,dt$. The function Φ_ω

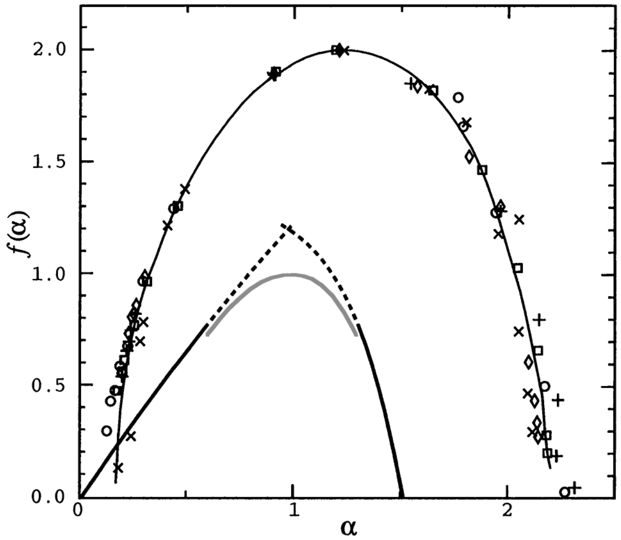

FIG. 1. The multifractal spectrum of passive scalar dissipation field in turbulent jet [1] (reproduced with permission). Bold lines indicate the curve predicted by Eqs. (13), (14), and (17). The following numerical values have been used: $u \approx 0.15$, $D \approx 0.6$, $r_0 \approx 0.01$, $t \approx 40$. Dashed lines correspond to regions, where the inequalities in (13) are not satisfied; the transition between the two asymptotic branches is sketched by the gray line. The experimental data describe 2D cross sections of the 3D dissipation field; the analytic curve is obtained for one-dimensional cross sections of the 2D field.

is expressed via Hankel functions, the order of which is $\sqrt{4D\omega + u^2}/2D$. The Laplace transform tends to infinity for $\omega \to -u^2/4D$; therefore we conclude that the asymptotic solution is

$$\Phi \to i\exp(-u^2t/4D)H_0^{(1)}(ik\sqrt{2\kappa/D})k^{u/2D}, \tag{18}$$

where $H^{(1)}$ is the Hankel function of the first kind. The exponential decay, independent of molecular diffusivity, is in full agreement with the earlier results [7,9].

Finally, we consider the power spectrum of dye in the case of a steady source, when the evolution of the tracer concentration is defined by Eqs. (1) and (3). Again, we model the process by Eq. (10), but the initial condition is to be substituted by boundary condition

$$\lambda(0,t) = 1\,, \tag{19}$$

corresponding to the statistically stationary source (3). We are looking for the stationary solution of the system (10) and (19). This is given again by the Hankel function:

$$\Phi = AH_{u/2D}^{(1)}(ik\sqrt{2\kappa/D})k^{u/2D}, \tag{20}$$

where $A = [H_{u/2D}^{(1)}(i\sqrt{2\kappa/D})]^{-1}$. Using asymptotic expansion of the Hankel function, at $k \gg \sqrt{D/2\kappa}$ we have $\Phi \to k^{(u/D-1)/2}\exp(-k\sqrt{2\kappa/D})$, implying an exponential tail as predicted by Kraichnan [6]; however, at the intermediate values of the argument a clearly different behavior is predicted. Expression (20) provides an excellent fit with the experimental data obtained with a magnetically forced two-dimensional fluid [8], yielding $u \approx 0.21$, $u/D \approx 0.24$, $\kappa/D \approx 5 \times 10^{-4}$.

In conclusion, we discuss the limitations and possible further developments of the model. While the type of external forcing $g(\boldsymbol{r},t)$ does not limit the applicability of Eq. (10) and defines just the boundary condition, serious difficulties will arise in the case of dye gradients and Eq. (7). Indeed, when dye is added, the gradients are affected by the coherence between the existing patterns and the newly added dye. For the same reason, here the molecular diffusivity cannot be taken into account as easily as in the case of power spectra.

Generalization of the model to the more realistic 3D flows is relatively simple; the main idea is to study the stretching of fluid surfaces. The velocity fields with turbulent power spectra and a lower viscous cutoff scale are also easily tractable. Then the main contribution to the stretching of fluid elements is made by the shortest-wavelength pulsations; thus, Eq. (7) can be directly applied (however, ρ^{-1} no longer defines the characteristic space scale of fluctuations). As an immediate application, the problem of atmospheric (and oceanic) pollution can be mentioned: a strongly localized source $g(\boldsymbol{r},t)$ leads to the formation of pollutant "blobs," the size of which is related to the local stretching factor. The approach devised for the multifractal analysis can be also extended to the problems of the kinematic magnetic dynamos [3] and the structure functions of scalar fields.

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