Superradiance Resonance Cavity Outside Rapidly Rotating Black Holes

Nils Andersson[1] and Kostas Glampedakis[2]

[1]Department of Mathematics, University of Southampton, Southampton SO17 1BJ, United Kingdom

[2]Department of Physics and Astronomy, Cardiff University, Cardiff CF2 3YB, United Kingdom

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We discuss the late-time behavior of a dynamically perturbed Kerr black hole. We present analytic results for near-extreme Kerr black holes that show that the large number of virtually undamped quasinormal modes that exist for nonzero values of the azimuthal eigenvalue m combine in such a way that the field oscillates with an amplitude that decays as $1/t$ at late times. This prediction is verified using numerical time evolutions of the Teukolsky equation. We argue that the observed behavior can be understood in terms of the presence of a "superradiance resonance cavity" immediately outside the black hole, and discuss whether it may be relevant for astrophysical black holes.

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Our understanding of the generic response of a black hole to dynamic perturbations is based on seminal work from 30 years ago. Exponentially damped quasinormal-mode (QNM) ringing was first observed in numerical experiments by Vishveshwara [1], and the subsequent late-time power-law falloff (that all perturbative fields decay as t^{-2l-3} in the Schwarzschild geometry) was discovered by Price [2]. A considerable body of work has since established the importance of these two phenomena for black-hole physics. We now know that most black-hole signals are dominated by the slowest damped QNMs, and many reliable methods for investigating these modes have been developed [3]. The nature of the late-time tail has also been studied in great detail. In particular, it has been established that it is a generic effect independent of the presence of an event horizon: The tail arises from backscattering off of the weak gravitational potential in far zone [4]. However, the fact that our understanding has reached a mature level does not mean that no problems remain in this field. A few years ago, the quasinormal modes had been calculated also for Kerr black holes [5], but there were no actual calculations demonstrating the presence of power-law tails. Neither were there any dynamical studies of rotating black holes. Several recent developments have served to change this situation and improve our understanding of dynamical rotating black holes. Of particular relevance has been an effort to develop a reliable framework for perturbative time evolutions of Kerr black holes [6]. There have also been recent efforts to analytically approximate the late-time power-law tails for Kerr black holes [7,8]. Furthermore, numerical relativity is now reaching a stage where fully nonlinear studies of spinning black holes are feasible [9].

Kerr black-hole spectroscopy.—With the likely advent of gravitational-wave astronomy only a few years away, the onus is on theorists to provide detailed predictions of the many scenarios that may lead to detectable gravitational waves. In this context, the question of whether we can realistically hope to do "black-hole spectroscopy" by detecting QNM signals and inverting them to infer the black-hole mass and angular momentum is highly relevant [10]. For slowly rotating black holes this presents a serious challenge. Using standard results one can readily estimate that the effective gravitational-wave amplitude for QNMs is (cf. similar estimates for pulsating stars [11])

$$h_{\mathrm{eff}} \approx 4.2 \times 10^{-24}\left(\frac{\delta}{10^{-6}}\right)^{1/2}\left(\frac{M}{M_{\odot}}\right)\left(\frac{15\ \mathrm{Mpc}}{r}\right), \quad (1)$$

where δ is the radiated energy as a fraction of the black-hole mass M. The frequency of the radiation depends on the black-hole mass as $f \approx 12(M_{\odot}/M)$kHz. Given these relations, and recalling the estimated sensitivity of the generation of detectors that is under construction, the detection of QNM signals from slowly rotating solar-mass black holes seems rather unlikely. It is, however, worth pointing out that the situation will be more favorable for low-frequency signals from supramassive black holes in galactic nuclei and detection with LISA, the space-based interferometric gravitational-wave antenna. It is also interesting to note recent suggestions that "middle weight" black holes, in the range $(100\text{–}1000)M_{\odot}$, may exist [12]. For such black holes the most important QNMs would radiate at frequencies where the new generation of ground based detectors reach their peak sensitivity. If there are indeed such black holes out there we may hope to take their fingerprints in the future.

It has been suggested that QNM signals from rapidly rotating black holes would be easier to detect. This belief is based on the fact that some QNMs become very long lived as $a \to M$ (where $0 \leq a \leq M$ is the rotation parameter of the black hole). In fact, mode calculations predict the existence of an infinite set of essentially undamped modes in the extreme Kerr limit [5]. The available investigations into the detectability of QNM signals have focused on the slow damping of these modes [10]. It has been shown that the decreased damping of the modes may increase the detectability considerably. However, these results have to be interpreted with some caution. What has been shown is the (anticipated) effect that a slower damped mode is easier to detect than a short-lived one, *provided that the modes are excited to a comparable amplitude*. This is a rather subtle issue that is closely related to the question of whether it is easier to excite a slowly damped QNM than a short-lived one. Intuitively, one might expect this not to be the case. In similar physical situations the buildup of energy in a long-lived resonant mode takes place on a time scale similar to the eventual mode damping. Thus it ought to be very difficult to excite a QNM that has characteristic damping several times longer than the dynamical time scale of the excitation process. This argument suggests that the amplitude of each long-lived mode ought to vanish in the limit $a \to M$ when the e-folding time of the mode increases dramatically [13]. In view of this it would seem rather dubious to conclude that the detectability of a QNM signal actually improves as $a \to M$. All may not be lost, however, because even if each individual QNM has an infinitesimal amplitude for rapidly spinning black holes a large number of modes approach the same limiting frequency as $a \to M$. These modes may interfere constructively to give a considerable signal [14].

A surprising analytic result.—We want to assess the change in "detectability" of the QNMs as $a \to M$, i.e., as we approach the extreme Kerr black hole case. As a suitable model problem, we consider a massless scalar field. As is well known, the equation that governs such a field (which follows immediately from $\Box\Phi = 0$) is similar to the master equation for both electromagnetic and gravitational perturbations of a rotating black hole that was first derived by Teukolsky [15]. In the following, we briefly outline our calculation and discuss the main results. A more exhaustive discussion will be presented elsewhere [16]. We use standard Boyer-Lindquist coordinates, and approach the QNM problem in the frequency domain (obtained via the integral transform used in [17]). Furthermore, we use the symmetry of the problem to separate the dependence on the azimuthal angle φ. In essence, we are using a decomposition:

$$\Phi = \frac{e^{im\varphi}}{2\pi} \sum_{l=0}^{\infty} \int_{-\infty}^{+\infty} \frac{R_{lm}(\omega, r)}{\sqrt{r^2 + a^2}} S_{lm}(\omega, \theta) e^{-i\omega t}\, d\omega \,. \tag{2}$$

It should be noted that the rotation of the black hole couples the various l multipoles through the (frequency dependent) spheroidal angular functions S_{lm} [15].

In direct analogy with the Schwarzschild case [17] the initial value problem for the scalar field can be solved using a Green's function constructed from solutions to the homogeneous radial differential equation for $R_{lm}(\omega, r)$. One of the required solutions that satisfies the causal condition at the event horizon, $r_+ = M + \sqrt{M^2 - a^2}$, has asymptotic behavior,

$$R_{lm}^{\text{in}} \sim \begin{cases} e^{-ikr_*} & \text{as } r \to r_+ \,, \\ A_{\text{out}} e^{i\omega r_*} + A_{\text{in}} e^{-i\omega r_*} & \text{as } r \to +\infty \,, \end{cases} \tag{3}$$

where

$$k = \omega - \frac{ma}{2Mr_+} = \omega - m\omega_+ \,, \tag{4}$$

ω_+ is the angular velocity of the event horizon, and r_* is the tortoise coordinate. It is useful to recall that a monochromatic wave is superradiant if it has frequency in the range $0 < \omega < m\omega_+$ [15].

A QNM is defined as a frequency ω_n at which $A_{\text{in}} = 0$. Assuming that $A_{\text{in}} \approx (\omega - \omega_n)\alpha_n$ close to $\omega = \omega_n$ we can deduce (via the residue theorem) that the contribution from each such mode to the evolution of the scalar field is

$$\Phi_n(t, r, \theta) = \frac{A_{\text{out}}}{2\omega_n \alpha_n} e^{-i\omega_n(t - r_*)} \sum_{l=0}^{\infty} S_{lm}(\omega_n, \theta) I_{lm} \,, \tag{5}$$

where $I_{lm}(\omega_n, r)$ is a complicated expression that depends on the details of the initial data (here assumed to have support only far away from the black hole); cf. [16,17].

Let us now focus on the case of nearly extreme Kerr black holes, i.e., on the case $a \approx M$. Then we can use an approximation due to Teukolsky and Press [15] that suggests that there will exist an infinite set of QNMs that can be approximated by [14,18]

$$\omega_n M \approx \frac{m}{2} - \frac{1}{4m} e^{(\theta - 2n\pi)/2\delta} (\cos\varphi - i \sin\varphi) \,, \tag{6}$$

where δ, θ, and φ are positive constants (not to be confused with the coordinates), and n is an integer labeling the modes. It is easy to see that as $n \to \infty$ the modes become virtually undamped, and that they are located close to the upper limit of the superradiant frequency interval. That such a set of long-lived QNMs will exist agrees with other mode calculations [5,18]. Given the location of the QNMs we can extend the calculation to deduce also the form of the asymptotic amplitudes A_{out} and A_{in} (or rather, the coefficient α_n) for each ω_n. This enables us to approximate the contribution of each long-lived QNM to the field via (5). Doing this we find that the longest lived modes have exponentially small amplitudes. Thus we predict that the individual QNM will not in general be excited to a large amplitude, in agreement with our intuitive expectations. Expressing this result in terms of the effective amplitude of a corresponding gravitational-wave QNM, we would have

$$h_{\text{eff}} \sim \sqrt{\frac{\text{Re}\,\omega_n}{\text{Im}\,\omega_n}} \frac{A_{\text{out}}}{\alpha_n} \sim e^{-n\pi/2\delta}. \tag{7}$$

In other words, the assumption that the long-lived modes may be easier to detect than (say) their short-lived counterparts for slowly rotating black holes is cast in serious doubt. A recent, more detailed calculation of the QNM excitation coefficients for $a \le M$ supports this conclusion [16].

This does not, however, mean that the long-lived QNMs are without relevance. On the contrary, the fact that there is a large number of such modes has a very interesting consequence. After combining all the long-lived modes we find

$$\sum_{n=0}^{\infty} \frac{A_{\text{out}}}{\alpha_n} e^{-i\omega_n(t-r_*)} \sim \frac{e^{-im\omega_+ t}}{t} \quad \text{as } t \to \infty . \tag{8}$$

This is an unexpected result: It suggests that, when summed, the contribution from the slowly damped QNMs of a near extreme Kerr black hole corresponds to an oscillating signal whose magnitude falls off with time as a power law. Furthermore, the decay of this signal is considerably slower than the standard power-law tail. The decay of $1/t$ should be compared to the tail results of, for example, Ori and Barack [8] that suggest that $\Phi \sim t^{-l-|m|-3-q}$, where $q = 0$ for even $l + m$ and 1 for odd $l + m$ (derived only for nonextreme black holes). Hence, we predict that the oscillating QNM tail will dominate the late-time behavior of a perturbed near extreme Kerr black hole.

We should point out that an oscillating $1/t$ tail can also be deduced from the existence of an additional branch cut in the Green's function for an extreme Kerr black hole. This branch cut arises because the effective potential in the Teukolsky equation falls off as a power of r_* near the horizon in the extreme case (rather than exponentially as in the case $a \leq M$). An estimate of the effect of this cut leads to (8); see [16] for a detailed discussion. It is also worth mentioning that oscillating power laws are known to arise in standard scattering problems whenever the Green's function has higher order poles [19].

Numerical confirmation.—Our analytic result is obviously surprising. However, in view of the many approximations involved in the derivation of (8) considerable caution is warranted, and a confirmation of the analytic prediction is desirable. Fortunately, the recent effort to develop a framework for doing perturbative time evolutions for Kerr black holes [6] provides the means for testing our result. Hence, we have performed a set of evolutions (for various values of m) using the same scalar field code that was used to study superradiance in a dynamical context [6]. As initial data we have chosen a generic Gaussian pulse originally located far away from the black hole.

Our numerical evolution results can be succinctly summarized as follows (further details regarding, for example, numerical convergence will be discussed elsewhere [16]):

(i) For extreme Kerr black holes ($a = M$) the numerical evolutions show the predicted oscillating $1/t$ behavior for all $m \neq 0$; cf. Fig. 1.

(ii) For $a < M$ we recover the anticipated exponential falloff at late times; cf. [6]. Still, the emerging signal differs considerably from a single QNM oscillation at intermediate times for near extreme black holes. We are currently investigating this behavior in further detail.

(iii) For axisymmetric perturbations ($m = 0$) the numerical evolution recovers the standard power-law tail. For our particular choice of initial data (that contain the $l = 0$ multipole) the tail falls off as t^{-3}.

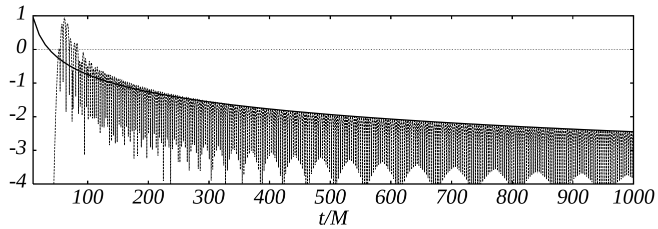

FIG. 1. A numerical evolution showing the late-time behavior of a scalar field in the geometry of a rapidly rotating black hole. We show (on a logarithmic scale) the field as viewed by an observer situated well away from the black hole for $a = M$. At late times the field falls off according to an oscillating power law with the amplitude decaying as $1/t$. The data correspond to a narrow Gaussian (initially centered at $r_* = 50M$) that hits the black hole, and the reflected wave is observed at $r_* = 10M$.

Our interpretation of these results is as follows. First, the numerical evolutions confirm the analytic prediction for extreme Kerr black holes, i.e., that the field will oscillate with an amplitude that decays as $1/t$ at very late times. Second, and more important physically, the numerical data suggest that the intermediate-to-late time behavior of a perturbed Kerr black hole is not well described by a single slowly damped QNM when $a < M$. This behavior must be investigated in greater detail in order to assess to what extent the late-time signals from a rapidly rotating black hole are detectable even though each individual QNM has a small amplitude. In particular, it is crucial to determine whether one will be able to infer the black hole parameters from a signal that contains a superposition of several slowly damped QNMs (perhaps using techniques similar to those introduced in [20]). It is also interesting to ask whether there exists a critical value of the rotation parameter a above which the new effect we have observed for the extreme case becomes relevant (recall that our approximate modes are relevant only for $a \approx M$). More detailed numerical work is needed to establish this, and to investigate the role of the new effect further.

A physical interpretation.—Given both the analytic prediction and the numerical confirmation for extreme Kerr black holes an intriguing picture emerges. Our results suggest the existence of a new phenomenon in black-hole physics, with relevance at late times. We recall that the QNMs are typically interpreted, in analogy with scattering resonances in quantum physics, as originating from waves that are temporarily trapped close to the peak of the curvature potential (corresponding to the unstable photon orbit at $r = 3M$ in the Schwarzschild space time), and that the late-time power-law tail arises because of backscattering off of the weak potential in the far zone. Can the present results be interpreted in a similar intuitive vein? We think they can, and propose the following explanation: Consider the fate of an essentially monochromatic wave that falls onto the black hole. Provided that the frequency is in the interval $0 < \omega < m\omega_+$ the wave will be superradiant. In effect, this means that a distant observer will see waves "emerging from the horizon" [cf. (3)], even though a local observer sees the waves crossing the event horizon (at r_+) [15]. This results in the scattered wave being amplified. In addition to this, one can establish that the effective potential has a peak outside the black hole (which is not immediately obvious since the potential is frequency dependent in the Kerr case) for a range of frequencies including the superradiant interval. The combination of the causal boundary condition at the horizon effectively corresponding to waves "coming out of the black hole" (according to a distant observer) and the presence of a potential peak leads to waves potentially being trapped in the region close to the horizon. In effect, there is a "superradiance resonance cavity" outside the black hole. Again according to a distant observer, waves can escape from this cavity only by leakage through the potential barrier to infinity. Since the superradiant amplification is strongest for frequencies close to $m\omega_+$, waves in the cavity experience a kind of parametric amplification and at very late times the dominant oscillation frequency ought to be $m\omega_+$. This is, of course, exactly what we have deduced from our analytic and numerical calculations. In the extreme black hole case leakage from the superradiance cavity leads to the observed $1/t$ decay. In the near extreme case, the existence of the cavity provides an intuitive explanation for the extremely slow damping of corotating QNMs with frequencies close to $m\omega_+$.

We have presented the results of an investigation into the late-time behavior of a perturbed Kerr black hole. An analytic calculation for the near extreme Kerr black hole case led to two important results. First, we deduced that even though some QNMs become very slowly damped as $a \to M$ these modes will not be easier to detect with a gravitational-wave detector than their rapidly damped Schwarzschild counterparts. Second, we arrived at the rather surprising prediction that the large number of virtually undamped QNMs that exist for each value of $m \neq 0$ combine in such a way that the field oscillates with an amplitude that decays as $1/t$ at late times. This decay is considerably slower than the standard power-law tail. The analytic prediction for near extreme black holes was then verified using numerical time evolutions of the Teukolsky equation. Whether this effect is of astrophysical relevance (recall that astrophysical black holes must have $a \leq 0.998M$ [21]) is an issue that requires further investigation, but it is possible that it will play a role at intermediate times for nonextreme black holes (for which the signal will be dominated by the slowest damped QNM at late times). Finally, we have proposed an intuitive explanation of the observed phenomenon: Waves of certain frequencies are effectively trapped in a "superradiance resonance cavity" immediately outside the black hole. In conclusion, we find these results tremendously exciting since they indicate the presence of a new phenomenon in black-hole physics.

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