Reflection and Transmission of Waves near the Localization Threshold

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A theory is presented for propagation of waves in bounded media near the mobility edge, based on the self-consistent theory for localization. It predicts a spatially inhomogeneous diffusion constant that leads to scale dependence in enhanced backscattering and transmission.

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Enhanced backscattering (EB) and localization of waves are two related subjects that have received a lot of attention in recent years [1–5]. They both find their origin in interference effects in multiple scattering of waves. EB with classical waves has elucidated the crucial role of reciprocity [6–8]. For electrons, interest has concentrated on weak localization effects, whose interpretation calls upon the same interference events that are observed directly in EB [9,10]. Recent experiments [4,11] call for a theory capable of describing reflection and transmission around the mobility edge. For open systems, the random-matrix theory [12] and the self-consistent (SC) theory of localization [13–15] have been developed. The first is nonperturbational and can even deal with fluctuations, but is restricted to quasi-1D systems as studied in microwave experiments [3].

The SC theory provides an implicit equation for the dynamic "ac" diffusion constant $D(\Omega)$ of the waves [15],

$$\frac{1}{D(\Omega,\mathbf{r})} = \frac{1}{D_B} + \frac{C_\Omega(\mathbf{r},\mathbf{r})}{\pi v_E \rho(k)}. \tag{1a}$$

The Boltzmann diffusion constant D_B is free from interference. The second term contains the average "return probability" $C(\mathbf{r},\mathbf{r})$ by constructive interference of reciprocal paths at position $\mathbf{r}$, which lowers the diffusion constant. We will ignore the difference between extinction length, scattering and Boltzmann transport mean-free path and represent all by ℓ. With v_E being the transport speed of light and k its wave number, we have (in 3D) the familiar relations $D_B = \frac{1}{3}v_E\ell$ [1], and $\rho(k) \approx k^2/\pi^2 v_E$ for the density of states per unit volume. k, ℓ, and v_E have been calculated near the localization threshold [16].

Without magnetic fields, the reciprocity principle requires the amount of constructive interference $C_\Omega(\mathbf{r},\mathbf{r})$ to be *exactly* equal to the amount of "incoherent" radiation returning at $\mathbf{r}$ [15,17]. C_Ω must thus obey the dynamic diffusion equation,

$$[-i\Omega - \nabla \cdot D(\Omega,\mathbf{r})\nabla]C_\Omega(\mathbf{r},\mathbf{r}') = \frac{4\pi}{\ell}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{1b}$$

The factor $4\pi/\ell$ appears when single scattering is adopted as a source for multiple scattering. In the rest of this paper we consider $\Omega = 0$, which describes stationary diffusion flow.

Equations (1a) and (1b), here formulated for three dimensions, must contain *the same* diffusion constant, and one seeks a "self-consistent" solution. In infinite media, $C(\mathbf{r},\mathbf{r}')$ is translationally invariant, so that the return probability $C(\mathbf{r},\mathbf{r})$ and diffusion constant $D(\mathbf{r})$ do not depend on $\mathbf{r}$. In reciprocal space $C(q) = 4\pi/\ell D q^2$, so that $C(\mathbf{r},\mathbf{r}) = \sum_q C(q) \sim \mu/D\ell^2$, assuming an upper cutoff $q_{\max} = \mu/\ell$. Hence,

$$D(\Omega = 0) = D_B\left(1 - \frac{\mu}{k^2\ell^2}\right). \tag{2}$$

The mobility edge, defined by $D = 0$, obeys an Ioffe-Regel–type criterion as derived microscopically by John *et al.* [18] and Economou *et al.* [19], and agrees with the numerical studies of the Anderson tight binding model [20,21]. The cutoff removes short wave paths from the return probability and influences the exact location of the mobility edge [19,22]. For $\mu = 1$ the mobility edge is located at $k\ell = 1$.

In finite media, translational symmetry is absent and Eq. (1a) requires that the diffusion constant $D(\Omega,\mathbf{r})$ be dependent on $\mathbf{r}$. This has not to our knowledge been considered before, but is unavoidable if one doesn't wish to give up the basic ingredients of the SC theory of localization: reciprocity and flux conservation. Previous work focused on a homogeneous but "scale-dependent" diffusivity kernel $D(\Omega,\mathbf{r}-\mathbf{r}')$, with Fourier transform $D(\Omega,\mathbf{q})$. Near the mobility edge, $D(\Omega,q) \sim q$ has been suggested [23–25]. The absence of such q dependence in the SC theory is sometimes considered a serious failure, in spite of its agreement with scaling arguments for the dynamic diffusivity $D(\Omega)$ [26] and its qualitative agreement with scaling theory [27], as shown by Wölfle and Vollhardt [15]. At the mobility edge, our local formulation of the SC theory predicts the scale dependence $D(z) \sim 1/z$, which leads to a transmission $T \sim 1/L^2$ of a slab with thickness L, and a rounding of the EB line shape. Both properties were previously interpreted as consequences of a scale-dependent diffusivity $D(q) \sim q$ [25]. Contrary to all other approaches, our local variant of the SC theory deals elegantly and explicitly with boundary conditions.

We consider stationary propagation, in a slab geometry of thickness L, and Fourier transform ($\mathbf{q}_{\|}$) the transverse coordinate. For $0 < z < L$, Eqs. (1a) and (1b) become

$$D(z)^{-1} = D_B^{-1} + \frac{2}{k^2 \ell} \int_0^{1/\ell} dq_{\|} q_{\|} C(z, z, q_{\|}), \tag{3a}$$

$$-\partial_z D(z) \partial_z C(z, z', q_{\|}) + D(z) q_{\|}^2 C(z, z', q_{\|}) = \frac{4\pi}{\ell} \delta(z - z'), \tag{3b}$$

$$C(0, z', q_{\|}) - z_e(0) \partial_z C(0, z', q_{\|}) = 0, \tag{3c}$$

$$C(L, z', q_{\|}) + z_e(L) \partial_z C(L, z', q_{\|}) = 0. \tag{3d}$$

The last two equations are the radiative boundary conditions at both sides of the slab, featuring the "extrapolation lengths" $z_e(0/L) \equiv 3 z_0 D(0/L)/v_E$ [28]. Berkovits and Kaveh [25] emphasized that flux conservation requires them to contain the diffusion constant D, including interference. In our theory, D is finite at the boundaries so that z_e is always nonzero, even in the localized regime, when D vanishes in the bulk; $z_0 = \frac{2}{3}$ corresponds to no internal reflection, and increases with increasing internal reflection. The value for z_0 in recent localization experiments [4,11] is estimated to be typically 10.

Equation (3b) is recognized as an ordinary, second-order differential equation with a source term. Without the latter, two independent solutions $f_{\pm}(z)$ exist with constant and nonzero Wronskian $W(q_{\|}) \equiv D(z) \times (f'_+ f_- - f'_- f_+)$. In a "quasi-1D" medium with transverse surface $A < \ell^2$, only the transverse mode $q_{\|} = 0$ contributes to Eq. (3a), with weight $1/A$. Equations (3a)–(3d) have analytical solutions with a similar scale dependence of the transmission as predicted by random matrix theory [29]. For a semi-infinite quasi-1D medium their solution is $D(z) = D(0) \exp(-2z/\xi)$, with $\xi = A \rho(k) v_E \ell$ the same localization length as found in random matrix theory [29]. Furthermore, $1/D(0) = 1/D_B + 2z_0/\xi$.

For the slab geometry, Eqs. (3a)–(3d) have been studied numerically. We first discuss the semi-infinite medium $L = \infty$. In that case $C(z, z', q_{\|})$ must be bounded at large z, z' so that, with $f_+(z)$ the growing solution, Eq. (3b) is solved for

$$C(z, z', q_{\|}) = \frac{f_+(z_<) f_-(z_>)}{W(q_{\|}) \ell / 4\pi} - P(q_{\|}) f_-(z) f_-(z'), \tag{4}$$

where $z_< = \min(z, z')$, $z_> = \max(z, z')$, and $P(q_{\|})$ is determined by the boundary condition (3c) at $z = 0$. For the critical value $k\ell = 1$ we have compared the numerical solution to the simple algebraic form,

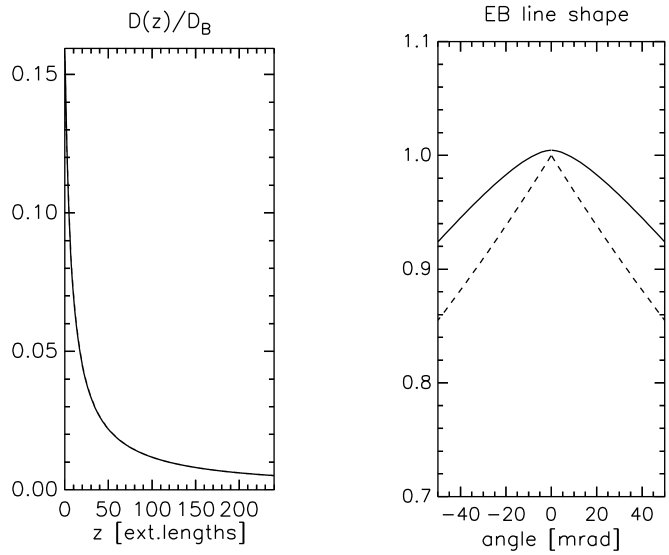

FIG. 1. Solution of the self-consistent equation at the mobility edge of a semi-infinite slab with internal reflection parameter $z_0 = 10$. Left panel: the diffusion constant $D(z)$ as a function of depth (in units of extinction lengths). Right panel: line profile in enhanced backscattering. The dashed line is the conventional cusp for $k\ell \gg 1$ using the same extrapolation length z_e.

$$D(z) = \frac{D(0)}{1 + z/\xi_c}, \tag{5}$$

with two free parameters $D(0)$ and ξ_c. The homogeneous solutions would then be

$$f_+(z) = (z + \xi_c) I_1(q_{\|}[z + \xi_c]); \quad f_-(z) = (z + \xi_c) K_1(q_{\|}[z + \xi_c]), \tag{6}$$

in terms of the modified Bessel functions I_1 and K_1 with Wronskian $W = D(0)\xi_c$ [30]. Equation (4) shows that $C(z, z, q_{\|})$ rises linearly in z for large z and that Eq. (3a) is asymptotically satisfied. The SC equation for $z = 0$ gives a relation between $D(0)$ and ξ_c. The remaining freedom in ξ_c was chosen to optimize self-consistency below 0.05%. (see Fig. 1 and Table I). Both ξ_c and $D(0)$ depend heavily on the parameter z_0 in the boundary condition.

The line shape $I_c(\theta)$ in EB can be obtained from $C(z, z', q_{\|})$ using standard methods [31] and is shown in Fig. 1. Insight is provided by the approximate formula

TABLE I. Solution $D(z) = D(0)/(1 + z/\xi_c)$ of the self-consistent equations (3a)–(3d) at the mobility edge $k\ell = 1$ for a semi-infinite slab as a function of the parameter z_0 that controls internal reflection at the boundary. The middle column shows the relation $D(0) \sim 1/z_0$.

z_0	$D(0)/D_B$	ξ_c/ℓ
2/3	0.642	1.5
3	0.336	3
5	0.249	4
7	0.203	6
10	0.159	8
20	0.0968	25

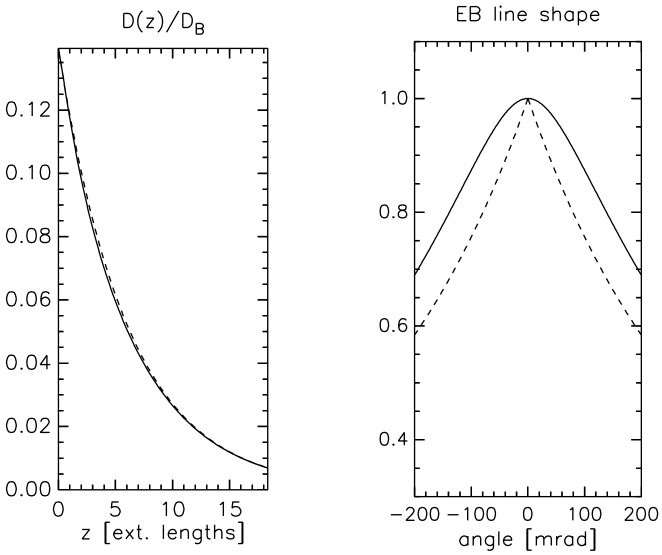

FIG. 2. Approximate solution of the self-consistent equations in the localized regime ($k\ell = 0.96$) of a semi-infinite slab with internal reflection parameter $z_0 = 10$, assuming an exponentially decaying diffusion constant. The dashed line in the left panel shows one iteration of the self-consistent equation. The dashed line in the right panel is the conventional cusp for $k\ell \gg 1$ using the same extrapolation length z_e.

$I_c(\theta) \approx C(z = \ell, z' = \ell, q_\parallel = 2k\sin\theta/2)$, used by Lagendijk *et al.* [28]. The line shape exhibits a logarithmic rounding $I_c(\theta) \sim 1 + z_e(0)\xi_c q_\parallel^2 \log(q_\parallel \xi_c)$ when $q_\parallel \xi \ll 1$, rather than the familiar cusp $I_c(\theta) \sim 1 - z_e|q_\parallel|$ [31]. Berkovits and Kaveh [25] predicted a rounding of the line shape on the basis of the nonlocal diffusion kernel $D(q)$.

The localized regime corresponds to $k\ell < 1$. As in quasi-1D media, we may assert the solution $D(z) = D(0)\exp(-2z/\xi)$, with ξ the localization length. We find $f_\pm(z) = \exp(-\lambda_\pm z)$ with $\lambda_\pm = 1/\xi \pm \sqrt{q_\parallel^2 + 1/\xi^2}$, and Wronskian $W = 2D(0)\sqrt{q_\parallel^2 + 1/\xi^2}$. The SC equation (3a) is satisfied for $z \gg \xi$ if $\xi/\ell = 2(k\ell)^2/[1 - (k\ell)^4]$, whereas the SC equation at $z = 0$ provides $D(0)$. Figure 2 shows the above exponential ansatz for D to be satisfactory for $z_0 = 10$, but for smaller internal reflection we found less agreement. The EB line shape is approximately given by

$$I_c(\theta) \approx \frac{1}{1 - z_e(0)/\xi + z_e(0)\sqrt{q_\parallel^2 + 1/\xi^2}}, \qquad (7)$$

i.e., an analytical rounding for $\theta < 1/k\xi$ (Fig. 2). This EB line shape is reminiscent of an absorbing semi-infinite medium in the delocalized regime, a case that must be excluded experimentally [11].

Our final subject is the length dependence of the total transmission $T(L)$ of a slab with length L. For a point source close to the boundary $z = 0$, the diffusion equation predicts

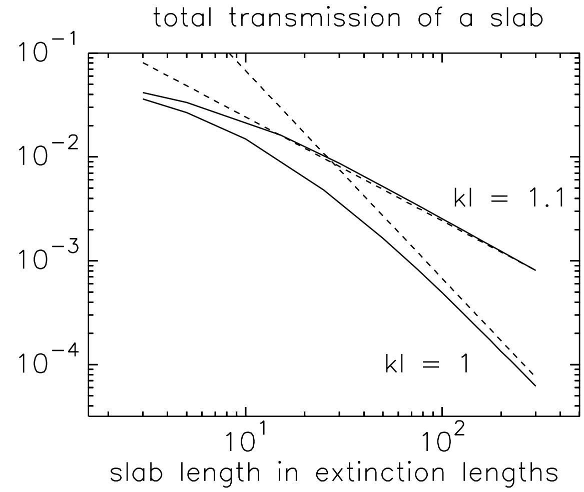

FIG. 3. Numerical solution of the self-consistent equations for a finite slab. The total transmission coefficient is displayed as a function of the slab length L, for the critical value $k\ell = 1$ and in the delocalized regime $k\ell = 1.1$. The dashed lines have slopes -2 and -1. We have adopted an internal reflection parameter $z_0 = 10$.

$$T(L) = z_0\ell\left(2z_0\ell + \int_0^L dz\,\frac{D_B}{D(z)}\right)^{-1}. \qquad (8)$$

The integral is proportional to the optical thickness of the slab. If $D(z)$ is constant, Eq. (8) reduces to the familiar result of radiative transfer with internal reflection [32], with the $1/L$ scaling. For a very long slab we expect the solution $D_\infty(z)$ for a semi-infinite medium to be relevant. More precisely, $D(z) \approx D_\infty(\frac{1}{2}L - |\frac{1}{2}L - z|)$. Equation (8) gives

$$T(L) \to \begin{cases} 4z_0[D_\infty(0)/D_B](\xi_c/\ell) \times (\ell/L)^2, & k\ell = 1, \\ z_0[D_\infty(0)/D_B](\ell/\xi) \times \exp(-L/\xi), & k\ell < 1. \end{cases} \qquad (9)$$

This scale dependence agrees with scaling theory [15,33], but has large and precise prefactors. (2.6 for $k\ell = 1$ and $z_0 = \frac{2}{3}$, and increasing with z_0). In Figure 3 we compare these results to the numerical solutions of Eqs. (3a)–(3d). The $1/L^2$ law predicted at the mobility edge is seen to disappear rapidly in the delocalized regime $k\ell > 1$. It has been reported by Garcia and Genack [2] and Wiersma *et al.* [4].

In summary, we have developed a theory for wave propagation in finite media near the mobility edge, adopting a local diffusion picture. Globally, scale-dependent diffusion is seen to emerge that captures quantitatively known scaling properties in transmission and enhanced backscattering.

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