Entanglement Teleportation via Werner States

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Transfer of entanglement and information is studied for quantum teleportation of an unknown entangled state through noisy quantum channels. We find that the quantum entanglement of the unknown state can be lost during the teleportation even when the channel is quantum correlated. We introduce a fundamental parameter of *correlation information* which dissipates *linearly* during the teleportation through the noisy channel. Analyzing the transfer of correlation information, we show that the purity of the initial state is important in determining the entanglement of the replica state.

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The nonlocal property of quantum mechanics enables a striking phenomenon called quantum teleportation. By quantum teleportation an unknown quantum state is destroyed at a sending place while its perfect replica state appears at a remote place via dual quantum and classical channels [1,2]. For the perfect quantum teleportation, a maximally entangled state, e.g., a singlet state, is required for the quantum channel. However, the decoherence effects due to the environment make the pure entangled state into a statistical mixture and degrade quantum entanglement in the real world. Popescu [3] studied the quantum teleportation with the mixed quantum channel and found that, even when the channel is not maximally entangled, it has fidelity better than any classical communication protocol. For a practical purpose, a purification scheme may be applied to the noisy channel state before teleportation [4–6].

Earlier studies have been confined to the teleportation of single-body quantum states: quantum teleportation of two-level states [1], *N*-dimensional states [7], and continuous variables [8,9]. In this Letter, we are interested in teleportation of two-body entangled quantum states, especially regarding the effects of the noisy environment. Direct transmission of an entangled state was considered in a noisy environment [10]. A possibility to copy pure entangled states was studied [11]. Extending the argument of the single-body teleportation we can easily show that an entangled *N*-body state can be perfectly teleported using the *N* maximally entangled pairs for the quantum channel. However, for the noisy channel, it is important and nontrivial to know how much the entanglement is transferred to the replica state and how close the replica state is to the original unknown state, depending on the entanglement of the unknown state and channel state.

Bennett *et al.* [1] argued that teleportation is a linear operation for the perfect quantum channel so that it would also work with mixed original states and could be extended to what is now called entanglement swapping [12]. We rigorously found that teleportation is linear even for the mixed channel, considering the maximization of the average fidelity [13]. With the property of the linearity, one

may *conjecture* that quantum teleportation preserves the nature of quantum correlation in the unknown entangled state if the channel is quantum mechanically correlated. We investigate this conjecture.

In this Letter, the original unknown state is assumed to be in an entangled two-body pure spin-1/2 state and the noisy quantum channel will be represented by a Werner state [14]. We define the measure of entanglement for the two spin-1/2 system and study the transfer of entanglement in the teleportation. We find that for the quantum channel there is a *critical value of minimum entanglement* required to teleport quantum entanglement. This minimum entanglement is understood by considering the transfer of entanglement and correlation information. The newly defined correlation information, which dissipates *linearly* during the teleportation through the noisy channel, is related to quantum entanglement for a pure state, and may also be related to classical correlation for a mixed state. Analyzing the transfer of correlation information, it is shown that the *purity of the initial state* is important in determining the entanglement of the replica state.

Before considering the entanglement teleportation protocol, we define a measure of entanglement. Consider a density matrix $\hat{\rho}$ and its partial transposition $\hat{\sigma} = \hat{\rho}^{T_2}$ for a two spin-1/2 system. The density matrix $\hat{\rho}$ is inseparable if and only if $\hat{\sigma}$ has any negative eigenvalues [15,16]. The measure of entanglement $\mathcal{E}(\hat{\rho})$ is then defined by

$$\mathcal{E}(\hat{\rho}) = -2 \sum_i \lambda_i^- , \qquad (1)$$

where λ_i^- is a negative eigenvalue of $\hat{\sigma}$. It is straightforward to prove that $\mathcal{E}(\hat{\rho})$ satisfies the necessary conditions required for every measure of entanglement [13,17].

The entanglement teleportation is schematically plotted in Fig. 1. The sender's unknown state $\hat{\rho}_{12}$ is prepared by the source S. Two *independent* EPR pairs are generated from E (one pair numbered 3 and 5, the other pair numbered 4 and 6 in Fig. 1). When a noisy environment is considered, its effects are attributed to the quantum channels, and the perfect EPR pair becomes mixed. By applying random SU(2) operations locally to both members of a pair a
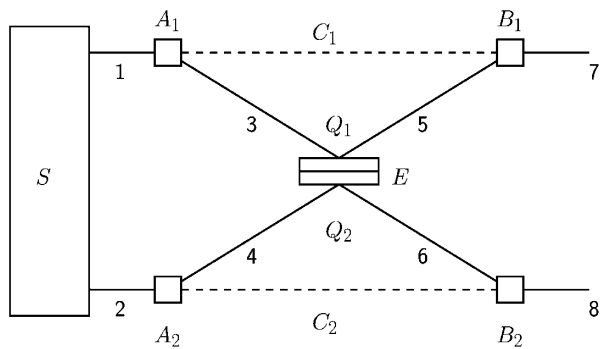
FIG. 1. Schematic drawing of entanglement teleportation. An unknown quantum entangled state is generated by the source S, and its particles are distributed separately into A_1 and A_2. The quantum channels Q_1 and Q_2 are represented by Werner states. The result of the Bell-state measurement at A_i ($i = 1, 2$) is transmitted through the classical channels C_i. The teleportation is completed by unitarily transforming at B_i according to the classical information.

general mixed two-body state becomes a highly symmetric Werner state which is SU(2) ⊗ SU(2) invariant [4,14]. For example, the quantum channel Q_1 is represented by the density matrix $\hat{w}_{35}$ of purity $(\Phi_{35} + 1)/2$ [14]:

$$\hat{w}_{35} = \frac{1}{4}\left(I \otimes I - \frac{2\Phi_{35} + 1}{3} \sum_n \sigma_n \otimes \sigma_n\right), \quad (2)$$

where σ_n is a Pauli matrix. The parameter Φ_{35} is related to the measure of entanglement $\mathcal{E}_{35}$, i.e., $\mathcal{E}_{35} \equiv \mathcal{E}(\hat{w}_{35}) = \max(0, \Phi_{35})$. To make our discussion simpler, we assume that the two independent quantum channels are equally entangled, i.e., $\mathcal{E}_{35} = \mathcal{E}_{46} \equiv \mathcal{E}_w$. In other words, $\Phi_{35} = \Phi_{46} \equiv \Phi_w$. This assumption can be justified as the two quantum channels are influenced by the same environment.

At A_i, a Bell-state measurement is performed on the particle i from S and one of the pair, $i + 2$, in the quantum channel Q_i. The Bell-state measurement at A_i is then represented by a family of projectors $\hat{P}_i^\alpha = |\Psi_i^\alpha\rangle\langle\Psi_i^\alpha|$ with $\alpha = 1, 2, 3, 4$, where $|\Psi_i^\alpha\rangle$ are the four possible Bell states [18]. The joint measurements at A_1 and A_2 project the total density matrix $\hat{\rho}$ onto the Bell states $|\Psi_1^\alpha\rangle$ and $|\Psi_2^\beta\rangle$, respectively, with the probability $P_{\alpha\beta} = \mathrm{Tr}\hat{P}_1^\alpha \hat{P}_2^\beta \hat{\rho}$. The probability $P_{\alpha\beta}$ is $1/16$ which is a characteristic of the Werner state. After receiving the two-bit information on the measurements through the classical channels C_1 and C_2, the unitary transformations $\hat{U}_1^\alpha$ and $\hat{U}_2^\beta$ are performed on the particles 5 and 6 accordingly.

By the unitary transformations, we reproduce the unknown state at B_1 and B_2 if the channel is maximally entangled. In choosing $\hat{U}_i^\alpha$, an important parameter to consider is the fidelity $\mathcal{F}$ defined as the distance between the unknown pure state $\hat{\rho}_{12}$ and the replica state $\hat{\rho}_{78}$: $\mathcal{F} = \mathrm{Tr}\hat{\rho}_{12}\hat{\rho}_{78}$. If $\hat{\rho}_{78} = \hat{\rho}_{12}$ then $\mathcal{F} = 1$, which shows that the replica is exactly the same as the unknown state and the teleportation has been perfect. The four unitary operations are given by the Pauli spin operators for the

singlet-state channel: $\hat{U}_i^1 = \hat{1}$, $\hat{U}_i^2 = \hat{\sigma}_x$, $\hat{U}_i^3 = \hat{\sigma}_y$, and $\hat{U}_i^4 = \hat{\sigma}_z$. For the Werner-state channel, we found that the same set of unitary operations $\hat{U}_i^\alpha$ are applied to maximize the fidelity [13]. The density matrices of both the original unknown state and the replica state can be written in the same form:

$$\hat{\rho} = \frac{1}{4}\left(I \otimes I + \vec{a} \cdot \vec{\sigma} \otimes I + I \otimes \vec{b} \cdot \vec{\sigma}\right.$$
$$\left. + \sum_{nm} c_{nm}\sigma_n \otimes \sigma_m\right). \quad (3)$$

The real vectors $\vec{a}$, $\vec{b}$, and real matrix c_{nm} of the replica state $\hat{\rho}_{78}$ are related to $\vec{a}_0$, $\vec{b}_0$, and c_{nm}^0 of the original state: $\vec{a} = (2\Phi_w + 1)\vec{a}_0/3$, $\vec{b} = (2\Phi_w + 1)\vec{b}_0/3$, and $c_{nm} = (2\Phi_w + 1)(2\Phi_w + 1)c_{nm}^0/9$.

The maximum fidelity $\mathcal{F}$ depends on the initial entanglement $\mathcal{E}_{12} = \mathcal{E}(\hat{\rho}_{12})$:

$$\mathcal{F} = \mathcal{F}^c + \mathcal{F}^q\mathcal{E}_{12}^2, \quad (4)$$

where $\mathcal{F}^c = (\mathcal{E}_w + 2)^2/9$, $\mathcal{F}^q = (2\mathcal{E}_w + 1)(\mathcal{E}_w - 1)/9$. When the unknown pure state is not entangled, i.e., $\mathcal{E}_{12} = 0$, the fidelity is just $\mathcal{F}^c$ which is the maximum fidelity for double teleportation of two independent particles [3,19]. For a given channel entanglement, the fidelity $\mathcal{F}$ decreases monotonously as the initial entanglement $\mathcal{E}_{12}$ increases because $\mathcal{F}^q \leq 0$. To obtain the same fidelity, the larger entangled channels are required for the larger entangled initial state. It implies that the entanglement is too fragile to teleport.

The measure of entanglement $\mathcal{E}_{78}$ for the replica state $\hat{\rho}_{78}$ is found using its definition (1) as

$$\mathcal{E}_{78} = \max\{0, \tfrac{1}{9}[(2\mathcal{E}_w^2 + 2\mathcal{E}_w - 4)$$
$$+ (1 + 2\mathcal{E}_w)^2\mathcal{E}_{12}]\}. \quad (5)$$

In Fig. 2, the entanglement $\mathcal{E}_{78}$ is plotted with respect to the entanglement $\mathcal{E}_{12}$ for the unknown state and $\mathcal{E}_w$ for the quantum channel. We find that $\mathcal{E}_{78}$ is nonzero, showing entanglement in the replica state only when $\mathcal{E}_w$ is larger than a critical value $\mathcal{E}_w^c \equiv (3 - \sqrt{2\mathcal{E}_{12} + 1})/(2\sqrt{2\mathcal{E}_{12} + 1})$. If the unknown state is maximally entangled with $\mathcal{E}_{12} = 1$, the quantum channel is required to have the entanglement larger than $\mathcal{E}_w^c \sim 0.3660$. It is remarkable that the entanglement teleportation has the critical value of minimum entanglement $\mathcal{E}_w^c \neq 0$ for the quantum channel to transfer any entanglement.

Brukner and Zeilinger [20] recently introduced a new measure of quantum information which is normalized to have n bits of information for n qubits. Based on their derivation, we define a measure of correlation information. The measure of total information for the density matrix $\hat{\rho}$ of the two spin-1/2 particles is $I(\hat{\rho}) = \frac{2}{3}(4\mathrm{Tr}\hat{\rho}^2 - 1)$, which may be decomposed into three parts. Each particle
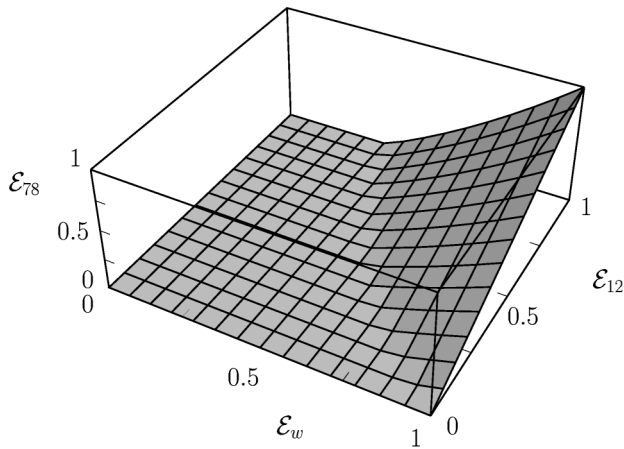
FIG. 2. Measure of entanglement $\mathcal{E}_{78}$ for the replica state $\hat{\rho}_{78}$ with respect to the entanglement $\mathcal{E}_{12}$ for the unknown pure state and $\mathcal{E}_w$ for the quantum channel.

has its own information corresponding to its marginal density matrix, which we call the *individual information.* The two particles can also share the *correlation information* which depends on how much they are correlated. The measure of individual information $I^a(\hat{\rho})$ for the particle a is

$$I^a(\hat{\rho}) = 2\,\mathrm{Tr}_a(\hat{\rho}_a)^2 - 1, \qquad (6)$$

where $\hat{\rho}_a = \mathrm{Tr}_b\hat{\rho}$ is the marginal density matrix for particle a. The measure of individual information $I^b(\hat{\rho})$ for particle b can be obtained analogously. If the total density matrix $\hat{\rho}$ is represented by $\hat{\rho} = \hat{\rho}_a \otimes \hat{\rho}_b$, the total system has no correlation. We define the measure of correlation information as [21]

$$I^c(\hat{\rho}) = I(\hat{\rho}) - I(\hat{\rho}_a \otimes \hat{\rho}_b). \qquad (7)$$

If there is no correlation between the two particles, the measure of total information is a mere sum of individual information. On the other hand, the total information is imposed only on the correlation information, $I = I^c$, if there is no individual information as for the singlet state. For a two-body spin-1/2 system, 1 bit is the maximum degree of each individual information while the correlation information can have a maximum of 2 bits.

The correlation information is in general contributed by quantum entanglement and classical correlation. When a pure entangled state is considered, its entanglement contributes to the whole of the correlation information. For a mixed state, on the other hand, the correlation information may also be due to classical correlation. For example, the Werner state with the entanglement $\mathcal{E}$ has the correlation information $I^c = \alpha + \beta\mathcal{E} + \gamma\mathcal{E}^2$ with constants α, β, and γ.

The entanglement teleportation transfers the correlation information $I^c_{12} \equiv I^c(\hat{\rho}_{12})$ of the unknown state $\hat{\rho}_{12}$ to the replica state $\hat{\rho}_{78}$. After straightforward algebra, we find that the transferred correlation information I^c_{78} is given by

$$I^c_{78} = \kappa^4 I^c_{12}, \qquad \kappa = \frac{2\mathcal{E}_w + 1}{3}, \qquad (8)$$

which shows that the correlation information dissipates *linearly* during the teleportation via the noisy quantum channel. When the channel is entangled, some correlation information, which can be classical or/and quantum mechanical, remains in the replica state. If the channel is entangled less than $\mathcal{E}^c_w$, I^c_{78} is totally determined by classical correlation. The reason why the teleportation does not necessarily transfer the entanglement to the replica state is that the correlation information for the replica state can be determined not only by quantum entanglement but also by classical correlation. We analyze it further as we separate the full teleportation into two partial teleportations of entanglement.

Consider a series of two partial teleportations of entanglement [22]. After the teleportation of particle 1 of the state $\hat{\rho}_{12}$, particle 2 of $\hat{\rho}_{72}$ is teleported and the final replica state is $\hat{\rho}_{78}$ in Fig. 1. We calculate the transfer of correlation information for the two teleportations

$$I^c_{72} = \kappa^2 I^c_{12}, \qquad I^c_{78} = \kappa^2 I^c_{72}. \qquad (9)$$

From these linear equations, we can easily recover Eq. (8). Now we investigate the dependence of correlation information on entanglement and classical correlation. For the entangled channel, $\mathcal{E}_w \neq 0$, the correlation information I^c_{72} can be written in terms of the entanglement $\mathcal{E}_{72}$ for $\hat{\rho}_{72}$:

$$I^c_{72} = 2\kappa^2\left(4 - 3\frac{\mathcal{E}_{72} + (1 - \mathcal{E}_w)}{\mathcal{E}_w(2 + \mathcal{E}_w)}\mathcal{E}_{72}\right)$$
$$\times \frac{\mathcal{E}_{72} + (1 - \mathcal{E}_w)}{\mathcal{E}_w(2 + \mathcal{E}_w)}\mathcal{E}_{72}, \qquad (10)$$

which shows that for $\mathcal{E}_w \neq 0$ the correlation information of the state $\hat{\rho}_{72}$ is due only to entanglement. The partial teleportation $\hat{\rho}_{12} \rightarrow \hat{\rho}_{72}$ transfers at least some of the initial entanglement as far as the channel is entangled. However, we have already seen that the final replica state $\hat{\rho}_{78}$ may include some classical correlation. The partial teleportation $\hat{\rho}_{72} \rightarrow \hat{\rho}_{78}$ may bring about no entanglement transfer. Why? The only difference between the two procedures is the purity of their initial states, as $\hat{\rho}_{12}$ is pure while $\hat{\rho}_{72}$ may be mixed. The purity of $\hat{\rho}_{72}$ is determined by the entanglement of the channel Q_1.

To analyze the importance of the initial purity for the entanglement transfer in partial teleportation, we release, for a while, the hereto assumption that both quantum channels have the same measure of entanglement. The entanglement $\mathcal{E}_{78}$ for the replica state then depends on the entanglement $\mathcal{E}_{46}$ of the quantum channel Q_2, and entanglement $\mathcal{E}_{72}$ and purity $\mathcal{P}_{72}$ of the state $\hat{\rho}_{72}$. The more Q_1 is entangled, the purer $\hat{\rho}_{72}$ is. We numerically calculate the dependence of entanglement $\mathcal{E}_{78}$ on the purity $\mathcal{P}_{72}$ of the intermediate state $\hat{\rho}_{72}$, as shown in Fig. 3. It clearly shows that the purity of the initial state determines the possibility of the entanglement transfer. This analysis can be
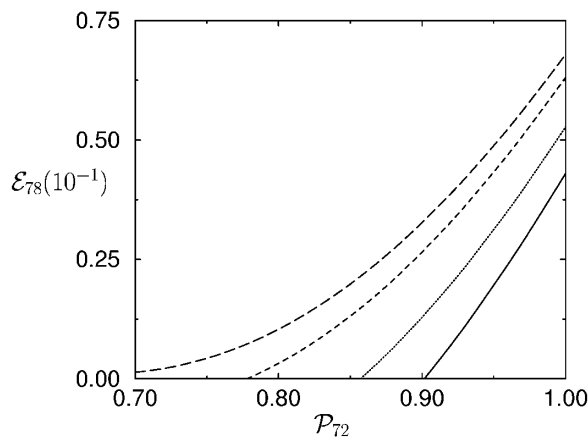
FIG. 3.   For the partial teleportation $\hat{\rho}_{72} \rightarrow \hat{\rho}_{78}$ with the channel entanglement $\mathcal{E}_{46} = 0.6$, the measure of entanglement $\mathcal{E}_{78}$ for the replica state is plotted against the purity $\mathcal{P}_{72}$. The entanglement $\mathcal{E}_{72} = 0.16$ (solid line), 0.18 (dotted line), 0.20 (dashed line), and 0.21 (long dashed line).

analogously applied to the other sequence of partial teleportations $\hat{\rho}_{12} \rightarrow \hat{\rho}_{18}$ and $\hat{\rho}_{18} \rightarrow \hat{\rho}_{78}$.

In conclusion, we investigated the effects of the noisy environment on the entanglement and information transfer in the entanglement teleportation. The introduction of the measures of entanglement and correlation information enables us to analyze intrinsic properties of the entanglement teleportation. We found that the teleportation always transfers the correlation information which dissipates linearly through the impure quantum channel. On the other hand, the entanglement transfer is not always possible. The analysis of partial teleportation shows that the purity of an initial state determines the possibility of the entanglement transfer. We explained this nontrivial feature by showing that a mixed state can have simultaneously quantum and classical correlations. Our studies on the entanglement transfer in the noisy environment will contribute to the entanglement manipulation, one of the basic schemes in quantum information theory.

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