New Types of Solitary Wave Solutions for the Higher Order Nonlinear Schrödinger Equation

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We present new types of solitary wave solutions for the higher order nonlinear Schrödinger (HNLS) equation describing propagation of femtosecond light pulses in an optical fiber under certain parametric conditions. Unlike the reported solitary wave solutions of the HNLS equation, the novel ones can describe bright and dark solitary wave properties in the same expressions and their amplitude may approach nonzero when the time variable approaches infinity. In addition, such solutions cannot exist in the nonlinear Schrödinger equation. Furthermore, we investigate the stability of these solitary waves under some initial pertubations by employing the numerical simulation methods.

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Propagations of ultrashort light pulses in optical fibers are of particular interest because of their extensive applications to telecommunication and ultrafast signal-routing systems. The recent progress of research on all-optical soliton transmission systems has revealed that they can overcome the limitations on the speed and distance of linear wave transmission systems [1]. In such a system, the higher order effects such as the third order dispersion (TOD), the self-steepening, and the self-frequency shift become important if the pulses are shorter than 100 fs [2]. When compared with the group velocity dispersion (GVD), the TOD is normally negligible but produces significant effects of asymmetrical temporal broadening for the ultrashort pulses [3,4]. The self-steepening effect, which is accompanied by an optical shock at trailing edge, also leads to the asymmetrical spectral broadening of the pulses [5]. The self-frequency shift due to stimulated Raman scattering results in an increasing redshift in the pulse spectrum, in which the long wavelength components experience Raman gain at the expense of the short wavelength components [6,7].

For large channel handling capacity and for high speed it is necessary to transmit solitary waves at a high bit rate of ultrashort pulses. So it is very important that all higher order effects be considered in the propagation of femtosecond pulses. In contrast, GVD and self-phase modulation (SPM) produce symmetric broadening in the time and frequency domains, respectively, and counterbalance to propagate solitons under some conditions, i.e., bright and dark soliton solutions exist in anomalous and normal dispersion regions, respectively. Similarly, there can be some possibilities to have soliton propagation with all higher order effects which induce asymmetrical broadening.

Taking account of these higher order effects mentioned above, Kodama *et al.* [8,9] derived the higher order nonlinear Schrödinger (HNLS) equation, which describes the propagation of ultrashort light pulses in optical fibers. In recent years many authors have analyzed the HNLS equation from different points of view (e.g., Painlevé analysis, Hirota direct method, Ablowitz-Kaup-Newell-Segur (AKNS) method, inverse scattering transform, Darboux-Bäcklund transform, and conservation laws) and obtained some types of exact solutions such as optical shock and bright N soliton for special values of the parameters [10–13]. Particularly, there have recently been some articles giving bright and dark solitary wave solutions for arbitrary values of parameters in the HNLS equation [14–17]. However, for all bright soliton or solitary wave solutions mentioned above, they are solved under the zero boundary conditions.

In this Letter, we present three new types of solitary wave solutions for the HNLS equation describing propagation of femtosecond light pulses in an optical fiber under certain parametric conditions. As we will show, unlike the reported solitary wave solutions of the HNLS equation, the novel ones can describe both bright and dark solitary wave properties in the same expressions and their amplitudes do not approach zero when the time variable approaches infinity. Furthermore, we investigate the stability of these solitary waves under some initial perturbations by employing numerical simulation methods.

The governing envelope wave equation for ultrashort light pulse propagation takes the form [8,9]

$$E_z = i(\alpha_1 E_{tt} + \alpha_2 |E|^2 E) + \alpha_3 E_{ttt} + \alpha_4 (|E|^2 E)_t + \alpha_5 E(|E|^2)_t \,, \tag{1}$$

where E is the slowly varying envelope of the electric field, the subscripts z and t are the spatial and temporal partial derivatives in retard time coordinates, and α_1, α_2, α_3, α_4, and α_5 are the real parameters related to GVD, SPM, TOD, self-steepening, and self-frequency shift arising from stimulated Raman scattering (SRS), respectively.

For picosecond light pulses, the last three terms of Eq. (1) can be omitted, and Eq. (1) can reduce to the nonlinear Schrödinger (NLS) equation. The NLS equation includes only the GVD and the SPM well known in the fiber, and it admits bright and dark soliton-type pulse propagation in anomalous and normal dispersion regimes, respectively [18]. However, for femtosecond light pulses, whose duration is shorter than 100 fs, the last three terms are non-negligible and should be retained.

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Although Eq. (1) can be reduced to a two-parameter canonical form by employing scaling transformation and the analyzing processes can be simplified as in Ref. [14], in the following analysis we still keep it in the original formulation to label the physical effect of each term in Eq. (1).

Now we proceed with the analysis of Eq. (1) by separating $E(z,t)$ into the complex envelope function $A(z,t)$ and linear phase shift $\phi(z,t) = \kappa z - \Omega t$ according to $E(z,t) = A(z,t)\exp[i\phi(z,t)]$. Substituting the expression into Eq. (1) and removing the exponential term, we can rewrite Eq. (1) as

$$iA_z + ia_1A_t + a_2A_{tt} - i\alpha_3A_{ttt} + a_3|A|^2A - ia_4|A|^2A_t - ia_5A^2A_t^* - a_6A = 0, \tag{2}$$

where $a_1 = -2\alpha_1\Omega + 3\alpha_3\Omega^2$, $a_2 = \alpha_1 - 3\alpha_3\Omega$, $a_3 = \alpha_2 - \alpha_4\Omega$, $a_4 = 2\alpha_4 + \alpha_5$, $a_5 = \alpha_4 + \alpha_5$, and $a_6 = \kappa + \alpha_1\Omega^2 - \alpha_3\Omega^3$.

Here, unlike the general assumption given in Ref. [17], we keep the envelope function $A(z,t)$ in a complex form so as to contain a possible nonlinear phase shift (see the following analysis).

In the following we look for the solitary wave solutions whose asymptotic values are nonzero when the time variable approaches infinity ($|t| \to \infty$) and make the ansatz

$$A(z,t) = \{i\beta + \lambda\tanh[\eta(t-\chi z)] + i\rho\,\mathrm{sech}[\eta(t-\chi z)]\}, \tag{3}$$

where η and χ are the pulse width and shift of inverse group velocity, respectively. The amplitude of $A(z,t)$ is

$$|A(z,t)| = \{(\lambda^2+\beta^2) + 2\beta\rho\,\mathrm{sech}[\eta(t-\chi z)] + (\rho^2-\lambda^2)\,\mathrm{sech}^2[\eta(t-\chi z)]\}^{1/2}, \tag{4}$$

and its corresponding nonlinear phase shift $\phi(z,t)$ is in the form

$$\phi(z,t) = \arctan\left(\frac{\beta + \rho\,\mathrm{sech}[\eta(t-\chi z)]}{\lambda\tanh[\eta(t-\chi z)]}\right). \tag{5}$$

When $\beta = \lambda = 0$ or $\rho = 0$, the ansatz (3) can reduce to a bright or dark solitary wave form, respectively. However, for general cases, the ansatz can describe the features of both bright and dark solitary waves.

Substituting Eq. (3) into Eq. (2) and setting the coefficients for the independent terms containing independent combinations of hyperbolic functions equal to zero, one obtains nine independent equations. The resulting equations are

$$\lambda[6\alpha_3\eta^2 - (a_4+a_5)(\rho^2-\lambda^2)] = 0, \tag{6}$$

$$\rho[6\alpha_3\eta^2 - (a_4+a_5)(\rho^2-\lambda^2)] = 0, \tag{7}$$

$$\rho[-2a_2\eta^2 + a_3(\rho^2-\lambda^2) - 2a_4\beta\lambda\eta] = 0, \tag{8}$$

$$\lambda[-2a_2\eta^2 + a_3(\rho^2-\lambda^2)] - 2\beta\eta[a_4\rho^2 + a_5(\rho^2-\lambda^2)] = 0, \tag{9}$$

$$\lambda[-\chi\eta + a_1\eta - 4\alpha_3\eta^3 - a_4\eta(\lambda^2+\beta^2) - a_5\eta(\lambda^2-\beta^2-2\rho^2)] + a_3\beta(3\rho^2-\lambda^2) = 0, \tag{10}$$

$$\rho[\chi\eta - a_1\eta + \alpha_3\eta^3 - 2\lambda\beta a_3 + (a_4-a_5)\lambda^2\eta + (a_4+a_5)\beta^2\eta] = 0, \tag{11}$$

$$\rho[a_2\eta^2 + a_3(\lambda^2+3\beta^2) + 2a_5\beta\lambda\eta - a_6] = 0, \tag{12}$$

$$\lambda[a_3(\lambda^2+\beta^2) - a_6] = 0, \tag{13}$$

$$\beta[a_3(\lambda^2+\beta^2) - a_6] = 0, \tag{14}$$

Obviously, for the case of $\beta = \lambda = 0$ or $\rho = 0$, these nine equations can reduce to four or five equations, and one can obtain the corresponding bright or dark solitary wave solutions, respectively, as given by Refs. [14–17]. Furthermore, for the case of $\alpha_3 = a_4 = a_5 = 0$, the HNLS equation can reduce to the NLS equation. From these nine parametric equations we can see that they are not compatible. This means that it is impossible that such a solution (3) should exist for the NLS equation. For the HNLS equation, Eqs. (6)–(14) are compatible if we impose some restrictions on the parameters. We have found that there are three types of solitary wave solutions for Eq. (2) under the following parametric conditions.

(i) $3\alpha_2\alpha_3 = \alpha_1\alpha_4$ *and* $\alpha_4 + 2\alpha_5 = 0$.—In this case, the solution (3) can be written in the form

$$A(z,t) = \{\lambda\tanh[\eta(t-\chi z)] + i\rho\,\mathrm{sech}[\eta(t-\chi z)]\}, \tag{15}$$

and its intensity is given by

$$|A|^2 = \{\lambda^2 + (\rho^2-\lambda^2)\mathrm{sech}^2[\eta(t-\chi z)]\}, \tag{16}$$

where

$$\eta^2 = \frac{\alpha_4}{3\alpha_3}(\rho^2-\lambda^2), \tag{17}$$

$$\chi = -(\alpha_1\Omega + \alpha_4\lambda^2) - \alpha_3\eta^2. \tag{18}$$

$$\Omega = \alpha_2/\alpha_4, \tag{19}$$

$$\kappa = -\frac{2}{3}\,\frac{\alpha_1\alpha_2^2}{\alpha_4^2}, \tag{20}$$

From Eq. (17) we can see that the solution (15) describing a bright or dark solitary wave depends on the specific nonlinear and dispersive features of medium intensity, and the pulse width is related to the difference of maximum and minimum intensity, i.e., $|\rho^2 - \lambda^2|$. Concretely speaking, if $\alpha_3\alpha_4 > 0$ (<0), from Eq. (17) one must require $\rho^2 - \lambda^2 > 0$ (<0), and the solution (15) represents a brightlike (darklike) solitary wave. For the brightlike case, different from the reported solitary wave solutions of the HNLS equation, the present one has a pronounced "platform" underneath it under nonzero boundary conditions and its asymptotic value approaches λ as the time variable approaches infinity ($|t| \to \infty$).

When $\lambda \to 0$ or $\rho \to 0$, the solution (15) becomes a specific bright or dark solitary wave solution under zero boundary conditions, respectively.

(ii) $\alpha_3 = 0$ *and* $\alpha_4 + \alpha_5 = 0$.—In this case, the solution (3) can be written as

$$A(z,t) = \{i\beta + \lambda \tanh[\eta(t - \chi z)] + i\lambda \operatorname{sech}[\eta(t - \chi z)]\}, \tag{21}$$

and its intensity is given by

$$|A|^2 = \beta^2 + \lambda^2 + 2\beta\lambda \operatorname{sech}[\eta(t - \chi z)], \tag{22}$$

where

$$\eta = -\frac{\alpha_4}{\alpha_1}\beta\lambda, \tag{23}$$

$$\lambda^2 = 2\alpha_1(\alpha_4\Omega - \alpha_2)/\alpha_4^2, \tag{24}$$

$$\kappa = (\alpha_2 - \alpha_4\Omega)(\lambda^2 + \beta^2) - \alpha_1\Omega^2, \tag{25}$$

$$\chi = -(2\alpha_1\Omega + \alpha_4\beta^2). \tag{26}$$

From Eq. (21) one must choose the parameter of frequency shift Ω to satisfy $\alpha_1(\alpha_4\Omega - \alpha_2) > 0$ although Ω is a undetermined parameter. As in the first case, the pulse width is also related to the difference of maximum and minimum intensity, i.e., $|2\beta\lambda|$. However, unlike the first case, the solution (21) describing a brightlike or a darklike solitary wave does not depend on the specific features of medium intensity and is only dependent on the initial pulses, that is, if $\beta\lambda > 0$ (<0), the solution (21) represents a brightlike (darklike) solitary wave. This feature indicates that a bright and dark solitary wave pulse may combine together under certain conditions and propagate simultaneously in an optical fiber with a combined form. Therefore, we suggest that a shorter and more descriptive name for this case is a combined solitary wave.

(iii) $\alpha_1 = \alpha_3 = 3\alpha_4 + 2\alpha_5 = 0$.—In this case, the solution (3) can be written as

$$A(z,t) = \{i\beta + \lambda \tanh[\eta(t - \chi z)] + i\rho \operatorname{sech}[\eta(t - \chi z)]\}, \tag{27}$$

and its intensity is given by

$$|A|^2 = (\lambda^2 + \beta^2) + 2\beta\rho \operatorname{sech}[\eta(t - \chi z)] + (\rho^2 - \lambda^2)\operatorname{sech}^2[\eta(t - \chi z)]. \tag{28}$$

where

$$\eta = -\frac{(\alpha_2 - \alpha_4\Omega)\beta}{(\alpha_4 + \alpha_5)\lambda}, \tag{29}$$

$$\rho^2 = \lambda^2 + 2\beta^2, \tag{30}$$

$$\kappa = (\alpha_2 - \alpha_4\Omega)(\lambda^2 + \beta^2), \tag{31}$$

$$\chi = 0. \tag{32}$$

For the special case of $\lambda = 0$, from Eqs. (6)–(14), one can further reduce the envelope function of the electric field $E(z,t)$ to

$$E(z,t) = (\beta + \rho \operatorname{sech}\eta t)\exp[-i(\alpha_2/\alpha_4)t], \tag{33}$$

where the parameters β, ρ, and η are determined by the incident pulses. As shown in Fig. 1, if setting $\beta\rho < 0$ and $|\rho| > |\beta|$, the intensity of the solitary wave $|E(z,t)|^2$ is of a peculiar feature in that it takes the shape of W. Therefore, we suggest that a shorter and more descriptive name for this case is a W-shaped solitary wave.

These solitary wave solutions might be one of the possible explanations for a single solitonlike pulse shape with a pronounced platform underneath it as commented by Bullough attached to Ref. [19]. As previously pointed out, the properties of femtosecond pulses can be described by the HNLS equation and there is no solitonlike solution with a platform underneath it for the NLS equation. Therefore, the origin of this platform might come from the higher order nonlinear effects.

In order to analyze the stability of these solitary wave solutions we have made numerical evolutions for initial optical pulses under some perturbations (i.e., amplitude, random noises, and the slight violation of the parameter conditions). It is shown that these solitary waves are still stable after propagating a distance of twenty dispersion lengths. Detailed stability analyses are now under investigation.

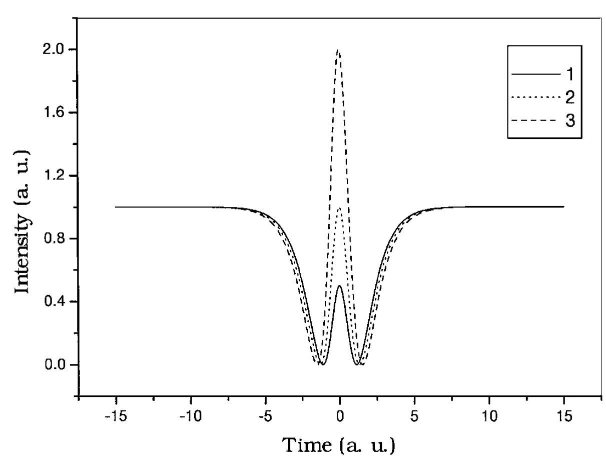

FIG. 1. Intensity of W-shaped solitary waves. The intensity is given by $|E(z,t)|^2 = \{\beta + \rho \operatorname{sech}[\eta(t - \chi z)]\}^2$. Curves 1, 2, and 3 are plotted at $z = 0$ by setting $\beta = 1$, $\eta = 1$, and $\rho = -(1 + \frac{\sqrt{2}}{2})$, $-(1 + \sqrt{2})$, and 2, respectively.

In conclusion, we have obtained new types of solitary wave solutions for the HNLS equation describing propagation of femtosecond light pulses in an optical fiber under certain parametric conditions. Unlike the reported solitary wave solutions of the HNLS equation, the novel ones can describe the properties of both bright and dark solitary waves in the same expressions and their amplitudes do not approach zero when the time variable approaches infinity. So these new types of solitary waves might be called combined solitary waves. These combined solitary waves may be decomposed into bright and dark ones, and this might be a potential application in communication systems and in the femtosecond laser systems which can produce bright and dark solitonlike pulses simultaneously. The numerical simulation shows that these new types of solitary waves are still stable under the perturbations of such things as amplitude, random noises, and the slight violation of parameter conditions after propagating a distance of twenty dispersion lengths.

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