

## Enhancement of Stochastic Resonance in Distributed Systems due to a Selective Coupling

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Recent massive numerical simulations have shown that the response of a “stochastic resonator” is enhanced as a consequence of spatial coupling. Similar results have been analytically obtained in a reaction-diffusion model, using *nonequilibrium potential* techniques. We now consider a field-dependent diffusivity and show that the *selectivity* of the coupling is more efficient for achieving stochastic-resonance enhancement than its overall value in the constant-diffusivity case.

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The phenomenon of *stochastic resonance* (SR)—namely, the *enhancement* of the output signal-to-noise ratio (SNR) caused by injection of an optimal amount of noise into a periodically driven nonlinear system—stands as one of the most puzzling and promising cooperative effects arising from the interplay between *deterministic* and *random* dynamics in a *nonlinear* system. The broad range of phenomena—indeed drawn from almost every field in scientific endeavor—for which this mechanism can offer an explanation has been put in evidence by many reviews and conference proceedings, Ref. [1] being the most recent and comprehensive one, from which one can scan the state of the art.

Most phenomena that could possibly be explained by SR occur in *extended* systems: for example, diverse experiments are being carried out to explore the role of SR in sensory and other biological functions [2] or in chemical systems [3]. Notwithstanding this fact, the overwhelming majority of the studies made up to now are based on zero-dimensional systems, while most of the features of this phenomenon that are peculiar to the case of extended systems—or *stochastically resonating media* (SRM)—still remain largely to be explored. Particularly interesting numerical simulations on arrays of coupled nonlinear oscillators have been recently reported [4], indicating that the *coupling* between these stochastic resonators enhances the response of the array, which exhibits moreover a higher degree of synchronization. This effect has its counterpart in the continuum, as a study on the overdamped continuous limit of a  $\phi^4$  field theory shows [5]. Recently—by exploiting the previous knowledge of the *nonequilibrium potential* (NEP) [6] for a bistable reaction-diffusion (RD) model [7]—one of us has shown analytically that the SNR increases with diffusivity in the range explored [8].

While considering a constant diffusion coefficient  $D$  is a standard approach, it is not the most general one: it is reasonable to expect that the reported enhancement in the SNR by the effect of diffusion could depend in a more detailed way on  $D$ . In this regard, see, for instance, Ref. [9]. In this Letter we consider the more realistic case

of a field-dependent diffusion coefficient  $D(\phi(x, t))$ , and show that it causes an *enhancement* of the SNR *still larger* than the one associated with a homogeneous increase of its amplitude.

The model under study—a one-dimensional, one-component RD model describing a system that undergoes an electrothermal instability [7]—can be regarded as the continuous limit of the coupled system studied by Lindner *et al.* [4]. The field  $\phi(x, t)$  might describe the (time-dependent) temperature profile in the “hot-spot model” of superconducting microbridges [7]. This model can be also regarded as a piecewise-linear version of the space-dependent Schlögl model for an autocatalytic chemical reaction, and that for the “ballast resistor,” describing the so-called “barretter effect” [10]. As a matter of fact, since in the ballast resistor the thermal conductivity is a function of the energy density, the resulting equation for the temperature field includes a temperature-dependent diffusion coefficient in a natural way [10]. Pointers to other contexts in which a description containing a field-dependent diffusivity becomes inescapable have been included in Refs. [11,12].

By adequate rescaling of the field, space-time variables, and parameters, we get a dimensionless time-evolution equation for the field  $\phi(x, t)$ ,

$$\partial_t \phi(x, t) = \partial_x (D(\phi) \partial_x \phi) + f(\phi), \quad (1)$$

where  $f(\phi) = -\phi + \theta(\phi - \phi_c)$ ,  $\theta(x)$  is Heaviside’s step function. All the effects of the parameters that keep the system away from equilibrium (such as the electric current in the electrothermal devices or some external reactant concentration in chemical models) are included in  $\phi_c$ . Moreover, since the value of the field  $\phi(x, t)$  corresponds in these models to the *deviations* with respect to, e.g., a reference temperature  $T_B > 0$  (the temperature of the bath) in the ballast resistor or to a reference concentration  $\rho_0$  in the Schlögl model, it is clear that—up to a given *strict* limit (i.e.,  $\phi = -T_B$  for the ballast resistor)—some negative values of  $\phi(x, t)$  are allowed.

As was done for the reaction term [7,8], a simple choice that retains, however, the qualitative features of the system is to consider the following dependence of the diffusion term on the field variable:

$$D(\phi) = D_0[1 + h\theta(\phi - \phi_c)]. \quad (2)$$

For simplicity, here we choose the same threshold  $\phi_c$  for the reaction term and the diffusion coefficient. The more general situation is left for a forthcoming paper [13].

We assume the system to be limited to a bounded domain  $x \in [-L, L]$  with Dirichlet boundary conditions at both ends, i.e.,  $\phi(\pm L, t) = 0$ . The piecewise-linear approximation of the reaction term in Eq. (1)—which mimicks a cubic polynomial—was chosen in order to find analytical expressions for its stationary spatially symmetric solutions. In addition to the trivial solution  $\phi_0(x) = 0$  (which is linearly stable and exists for the whole range of parameters) we find another linearly stable nonhomogeneous structure  $\phi_s(x)$ —presenting an excited central zone [where  $\phi_s(x) > \phi_c$ ] for  $-x_c \leq x \leq x_c$ —and a similar unstable structure  $\phi_u(x)$ , which exhibits a smaller excited central zone. The form of these patterns is analogous to what has been obtained in previous related works [7], as shown in Fig. 1. The difference is that in the present case  $d\phi/dx|_{x_c}$  is discontinuous and the area of the central zone depends on  $h$ .

The indicated patterns are extrema of the NEP, which—among other properties that we shall be using presently—is a Lyapunov functional for the *deterministic* system introduced thus far. In fact, the unstable pattern  $\phi_u(x)$  is a *saddle point* of this functional, separating the *attractors*  $\phi_0(x)$  and  $\phi_s(x)$  [7]. The notion of a NEP has been thoroughly studied, mainly by Graham and his collaborators [6]. Loosely speaking, it is an extension to nonequilibrium situations of the familiar notion of (equi-

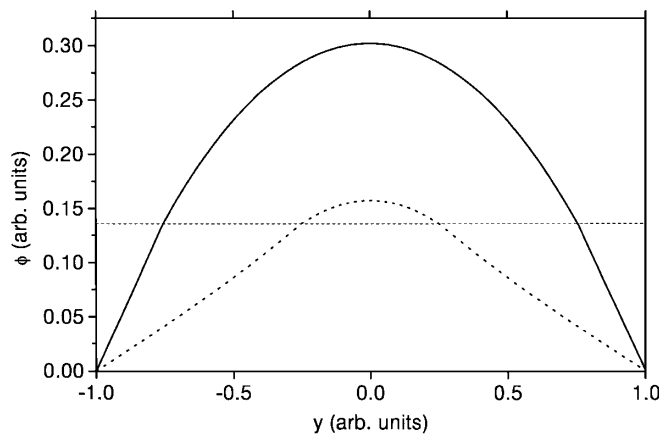


FIG. 1.  $\phi_s(y)$  is the *stable* pattern, and  $\phi_u(y)$  is the *unstable* one (similar in form but exhibiting a smaller excited central zone); they are extrema of the NEP:  $\phi_u(y)$  is a *saddle point* of this functional, separating the *attractors*  $\phi_0(y)$  and  $\phi_s(y)$  [7]. Here we indicate  $\phi_0(y)$  and  $\phi_s(y)$  with a solid line,  $\phi_u(y)$  with a dotted line, while the value of  $\phi_c$  is shown with a dashed line.

librium) *thermodynamic potential*. For the case of a field-dependent diffusion coefficient  $D(\phi(x, t))$  as described by Eq. (1), it reads [7,10]

$$\mathcal{F}[\phi] = \int_{-L}^{+L} \left\{ - \int_0^\phi D(\phi') f(\phi') d\phi' + \frac{1}{2} \left( D(\phi) \frac{\partial \phi}{\partial x} \right)^2 \right\} dx. \quad (3)$$

Given that  $\partial_t \phi = -[1/D(\phi)]\delta\mathcal{F}/\delta\phi$ , one finds  $\dot{\mathcal{F}} = -\int (\delta\mathcal{F}/\delta\phi)^2 dx \leq 0$ , thus warranting the Lyapunov-functional property. This NEP functional offers the possibility of studying not just the linear but also the nonlinear—in the case at hand, the *global*—stability of the patterns, following its changes as the parameters of the model are varied [7].

For a given threshold value  $\phi_c^*$ , both wells corresponding in a representation of the NEP to the linearly stable states have the same depth (i.e., both states are equally stable). Figure 2 shows the dependence of  $\mathcal{F}[\phi]$  on the parameter  $\phi_c$ . As in previous cases, we analyze only the neighborhood of  $\phi_c = \phi_c^*$  [8,14]. Here we also consider the neighborhood of  $h = 0$ , where the main trends of the effect can be captured.

Now, with the aim of studying SR, we introduce a weak signal that modulates the potential  $\mathcal{F}$  around the situation in which the two wells have the same depth. This is accomplished by allowing the parameter  $\phi_c$  to oscillate around  $\phi_c^*$ :  $\phi_c(t) = \phi_c^* + \delta\phi_c \cos(\Omega t + \varphi)$ , with  $\delta\phi_c \ll \phi_c(t)$ . We also introduce in Eq. (1) a fluctuating term  $\xi(x, t)$ , which we model (as is customary) as an additive Gaussian white-noise source with zero mean value and a correlation function  $\langle \xi(x, t)\xi(x', t') \rangle = 2\gamma\delta(t - t')\delta(x - x')$ , thus yielding a stochastic partial differential equation for the random field  $\phi(x, t)$ . The parameter  $\gamma$  denotes the noise strength [15].

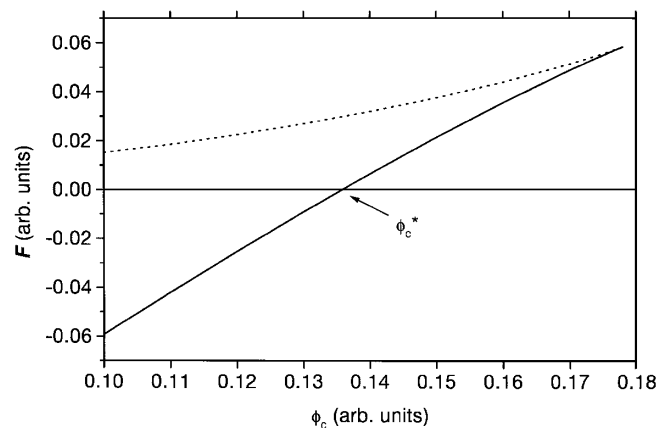


FIG. 2. Dependence on  $\phi_c$  of the value of  $F$  calculated for the patterns in Fig. 1. The stable branch  $F_s$  (indicated with a solid line) and the unstable one  $F_u$  (indicated with a dotted line) collapse at a critical value of  $\phi_c$ . At a certain value  $\phi_c = \phi_c^*$  (indicated by an arrow), the value of  $F_s$  becomes positive and  $\phi_s(x)$  becomes metastable.

As in previous works [7], we exploit a generalization to extended systems of the Kramers-like result for the evaluation of the decay time or “mean-first-passage time”  $\langle\tau\rangle$  [16,17]. Here, those results are extended to the case of field-dependent diffusivity, yielding [13]

$$\langle\tau\rangle = \tau_0 \exp\left\{\frac{\mathcal{W}[\phi, \phi_i]}{2\gamma}\right\}. \quad (4)$$

The functional  $\mathcal{W}[\phi, \phi_i]$  ( $\phi_i$  indicates the initial metastable state, which at each instant may be either  $\phi_0$  or  $\phi_s$ ), that is the solution of a Hamilton–Jacobi-like equation, has the following expression:

$$\mathcal{W}[\phi, \phi_i] = \int_{-L}^{+L} dx \left\{ \left[ \frac{D(\phi)}{2} \left( \frac{\partial\phi}{\partial x} \right)^2 - U(\phi) \right] - \left[ \frac{D(\phi_i)}{2} \left( \frac{\partial\phi_i}{\partial x} \right)^2 - U(\phi_i) \right] \right\}, \quad (5)$$

with  $U(\phi) = \int_0^\phi d\phi' f(\phi')$ . The prefactor  $\tau_0$  in Eq. (4) is essentially determined by the curvature of the NEP  $\mathcal{F}[\phi, \phi_c]$  at its extrema.

The calculation of the SNR proceeds, for the spatially extended problem, through the evaluation of the space-time correlation function  $\langle\phi(y, t)\phi(y', t')\rangle$ . To do that we use a simplified point of view, based on the two-state approach [18], which allows us to apply some known results almost directly. To proceed with the calculation of the correlation function, we need to evaluate the transition probabilities  $W_\pm \propto \langle\tau\rangle^{-1}$ , which appear in the associated master equation. For small  $\delta\phi_c$ ,

$$\mathcal{W}[\phi, \phi_i] \approx \mathcal{W}[\phi, \phi_i]_{\phi_c^*} + \delta\phi_c \left( \frac{\partial\mathcal{W}[\phi, \phi_i]}{\partial\phi_c} \right)_{\phi_c^*} \times \cos(\Omega t + \varphi).$$

By solving such a master equation up to first order in  $\delta\phi_c$ , it is possible to evaluate the correlation function. Its double Fourier transform, the generalized susceptibility  $S(\kappa, \omega)$ , factorizes in this approach, and the relevant term becomes a function of  $\omega$  only (the corresponding expressions are omitted, see [14] for details).

It is worth noting that many of the results exposed here (e.g., the profiles of the stationary patterns and the corresponding values of the NEP) are exact. The only approximations involved in the calculation of the SNR are the standard ones, namely, the Kramers-like expression in Eq. (4) and the two-level approximation used for the evaluation of the correlation function [18].

Using the definition from Ref. [18] for the SNR at the excitation frequency (here indicated by  $R$ ), the final result is

$$R \sim \left( \frac{\Lambda}{\tau_0\gamma} \right)^2 \exp(-\mathcal{W}[\phi, \phi_i]_{\phi_c^*}/\gamma), \quad (6)$$

where  $\Lambda = (d\mathcal{W}[\phi, \phi_i]/d\phi_c)_{\phi_c^*} \delta\phi_c$ , and  $\tau_0$  is given by the asymptotically dominant linear stability eigenval-

ues:  $\tau_0 = 2\pi(|\lambda^{\text{un}}| \overline{|\lambda^{\text{st}}|})^{-1/2}$  ( $\lambda^{\text{un}}$  is the unstable eigenvalue around  $\phi_u$ , and  $\overline{|\lambda^{\text{st}}|}$  is the average of the smallest eigenvalues around  $\phi_0$  and  $\phi_s$ ). Equation (6) is analogous to the results in zero-dimensional systems, but here  $\Lambda$ ,  $\tau_0$ , and  $\mathcal{W}[\phi, \phi_i]_{\phi_c^*}$  contain all the information regarding the spatially extended character of the system.

In Fig. 3 we depict the dependence of  $R$  on the noise intensity  $\gamma$ , for several (positive) values of  $h$ . These curves show the typical maximum that has become the fingerprint of the stochastic resonance phenomenon. Figure 4 is a plot of the value  $R_{\text{max}}$  of these maxima as a function of  $h$ . The dramatic increase of  $R_{\text{max}}$ , of several dB for a *small* positive variation of  $h$ , is apparent and shows the strong effect that the selective coupling (or field-dependent diffusivity) has on the response of the system.

It must be noted that the only two approximations made in order to render Eq. (6)—namely, the Kramers-like expression in Eq. (4) and the two-level approximation used for the evaluation of the correlation function [18]—break down for large positive values of  $h$  because, for increasing selectivity, the curves of  $\mathcal{F}[\phi]$  vs  $\phi_c$  in Fig. 2 shift towards the left, while the barrier separating the attractors at  $\phi_c^*$  tends to zero. This effect is basically the same as the one discussed in Ref. [8] in connection with the increase of the global diffusivity  $D_0$ , and also it is the physical reason why arbitrarily large enhancements of the SNR cannot be achieved by simply increasing  $D_0$  or  $h$ . It is also worth noting that, besides the two aforementioned approximations, all of the previous results (e.g., the profiles of the stationary patterns and the corresponding values of the nonequilibrium potential) are analytically exact.

The present prediction prompts one to devise experiments (for instance, through electronic setups) as well as numerical simulations, taking into account the indicated selective coupling. This result could be of relevance for technological applications such as signal detection and image recognition, as well as for the solution of some puzzles

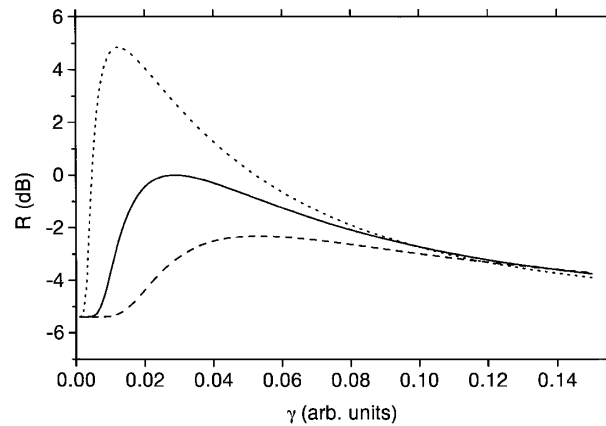


FIG. 3. SNR  $R$  as a function of the noise intensity  $\gamma$  [Eq. (6)], for three values of  $h$ :  $h = 0.0$  (solid line),  $-0.25$  (dashed line), and  $0.25$  (dotted line). We have fixed  $L = 1$ ,  $D_0 = 1$ ,  $\delta\phi_c = 0.01$ , and  $\Omega = 0.01$ .

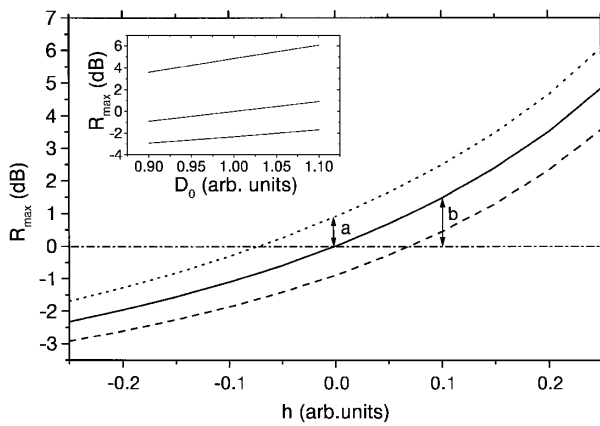


FIG. 4. Maximum  $R_{\max}$  of the SNR curve (Fig. 3) as a function of  $h$ , for three values of  $D_0$ :  $D_0 = 0.9$  (dashed line), 1 (solid line), and 1.1 (dotted line). The arrows  $a$  and  $b$  indicate the response gain due to a homogeneous increase of the coupling and to a selective one, respectively. The larger gain in the second case is apparent. The inset shows the dependence of  $R_{\max}$  on  $D_0$  for  $h = -0.25$  (lower line), 0, and 0.25 (upper line).

in biology (mammalian sensory systems, ionic channels in cells).

The present form of analysis is being extended to (the bistable regime of) multicomponent models of the activator-inhibitor-type since—in addition to their applications to systems of chemical (e.g., Bonhoffer–Van der Pol model) and biological (e.g., FitzHugh–Nagumo model) origins—these models are related to spatiotemporal synchronization problems [1,2,4]. An effective treatment of models of this type gives rise to a nonlocal coupling, which would compete with the nearest neighbor coupling  $D(\phi)$  presented here [14].

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