Fragmented and Single Condensate Ground States of Spin-1 Bose Gas

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We show that the ground state of a spin-1 Bose gas with an antiferromagnetic interaction is a fragmented condensate in uniform magnetic fields. The number fluctuations in each spin component change rapidly from being enormous (order N) to exceedingly small (order 1) as the magnetization of the system increases. A fragmented condensate can be turned into a single condensate state by magnetic field gradients. The conditions for existence and method of detecting fragmented states are presented.

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In this paper, we address the question of whether the ground state of a Bose gas with internal degrees of freedom is what one normally expects, i.e., a state with a single coherent Bose condensate. We show that this need not be the case and that experiments on optically trapped ^{23}Na spin-1 bosons can help to uncover many highly unusual yet fundamental properties of Bose systems with internal degrees of freedom.

In 1956, Penrose and Onsager [1] succeeded in generalizing the concept of Bose-Einstein condensation to interacting Bose systems. A system of N boson is considered Bose condensed if its single particle density matrix has a single macroscopic eigenvalue (i.e., of order N). The corresponding eigenfunction is identified as the quantum state macroscopically occupied. The possibility of a ground state with more than one macroscopic eigenvalue in its density matrix (referred to as a "fragmented" condensate) was considered by Nozieres and Saint James [2] in 1982. They also concluded that fragmentation cannot occur in a homogeneous scalar Bose gas with repulsive interactions. So far, the single condensate interpretation is in good agreement with experiments on magnetically trapped alkali atoms, which are effectively scalar bosons since their spins are frozen. Recently, optical trapping of a Bose condensate has become possible [3]. Since optical traps confine all spin states, the nature of the condensate will depend on the magnetic interaction between different spin states [4,5]. For ^{23}Na, which is a spin-1 Bose gas with antiferromagnetic interaction [4], the single spinor condensate picture (which is a mean field approximation) also appears to agree with experiments [6]. (Since the single condensate of the alkaki bosons is described by the usual coherent state, we shall from now on use the terms "single condensate state" and "coherent state" interchangeably.)

In a recent paper, however, Law, Pu, and Bigelow [7] pointed out that the ground state of the Hamiltonian in Refs. [4,5] in zero magnetic field is a spin singlet, with properties drastically different from those of coherent spinor condensates. They, however, did not discuss how their results could reconcile with the MIT experiments, which show that the coherent state picture works well at least in finite external magnetic fields [6]. In this paper, we show that the ground state of a spin-1 Bose gas can undergo dramatic changes in response to magnetic perturbations. We show that the singlet ground state in zero field is a fragmented condensate with anomalously large number fluctuations and thus has fragile stability. As the magnetization of the system increases, the singlet state quickly gives way to a much more generic fragmented state [essentially a Fock (or a number) state] with essentially zero fluctuations. The origin of fragmentation turns out to be spin conservation. As a result, by turning on a field gradient (which destroys spin conservation), a fragmented state can be deformed gradually toward coherent states as the strength of the field gradient increases. As we shall see, the Fock state and the coherent state have identical occupation numbers in each spin component, except that there is no phase coherence in the former case. As a result, these two states cannot be distinguished by density measurements such as the MIT experiment [6]. However, they can be told apart by the phase coherence of the coherent condensate, which allows Josephson tunneling between different spin components in the presence of magnetic field gradients (as we discuss later). In the following, we first consider a homogeneous Bose gas, where the "fragmentation" can be discussed most efficiently. Discussions on trapped gases will follow.

Homogeneous spin-1 Bose gas.—Consider a spin-1 Bose gas with Hamiltonian [4,5] $\hat{H} = \hat{h} + \hat{V}$, $\hat{h} = \int \hat{\psi}^{\dagger}_{\mu} h(\mathbf{x})_{\mu\nu} \hat{\psi}_{\nu}$, $h(\mathbf{x})_{\mu\nu} \equiv -\frac{\hbar^2}{2M}\nabla^2\delta_{\mu\nu} - \gamma \mathbf{B}\cdot\mathbf{S}_{\mu\nu}$, $\hat{V} = \frac{1}{2}\int \hat{\psi}^{\dagger}_{\alpha}\hat{\psi}^{\dagger}_{\mu}\hat{\psi}_{\nu}\hat{\psi}_{\beta}[c_0\delta_{\alpha\beta}\delta_{\mu\nu} + c_2\mathbf{S}_{\alpha\beta}\cdot\mathbf{S}_{\mu\nu}]$, where $c_0, c_2 > 0$, M and γ are the mass and gyromagnetic ratio of the boson, respectively, and $\mathbf{B}$ is a uniform magnetic field. The field operator $\hat{\psi}_{\mu}(\mathbf{x})$ ($\mu = \pm 1, 0$) can be expanded as $\hat{\psi}_{\mu}(\mathbf{x}) = \Omega^{-1/2}\sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mu}(\mathbf{k})$, where Ω is the volume of the system. For simplicity, we denote $a_{\mu}(\mathbf{k}=0)$ simply as a_{μ}. Denoting the part of $\hat{H}$ containing a_{μ} alone as $\hat{H}_o$ and the rest as $\hat{H}_{\text{ex}}$, ($\hat{H} = \hat{H}_o + \hat{H}_{\text{ex}}$), we have

$$\hat{H}_o = \frac{c_2}{2\Omega}(\mathbf{S}^2 - 2N) - \gamma \mathbf{B}\cdot\mathbf{S} + D, \quad (1)$$

where $\mathbf{S} = a_\mu^\dagger \mathbf{S}_{\mu\nu} a_\nu$, $N = a_\mu^\dagger a_\mu$, and $D = \frac{c_0}{2\Omega}[N^2 - N]$.

To find the ground state $|G\rangle$, we first find the ground state of $\hat{H}_o$ (denoted as $|F\rangle$) and then study condensate depletion effects due to $\hat{H}_{\rm ex}$. From Eq. (1), we see that $|F\rangle = |S^{\rm total} = S; S_z^{\rm total} = S\rangle$, where S is the *integer* that minimizes the energy $\langle H_o\rangle_F = (c_2/2\Omega)S(S+1) - \gamma BS - c_2\frac{N}{\Omega} + D$. In contrast, the optimum single condensate state in a magnetic field is

$$|C\rangle = \frac{1}{\sqrt{N!}}\left(\sqrt{\frac{N_1}{N}}\,a_1^\dagger + \sqrt{\frac{N_{-1}}{N}}\,a_{-1}^\dagger\right)^N |\text{vac}\rangle, \quad (2)$$

where $N_{\pm 1} = N(1 \pm y)/2$ and y is the optimal magnetization obtained by minimizing the energy $\langle \hat{H}_o\rangle_C = \frac{c_2}{2\Omega}N(N-1)y^2 - \gamma BNy + D$. It is easy to show that $y = S$ to order N^{-1} and that the difference $\langle \hat{H}_o\rangle_F - \langle \hat{H}_o\rangle_C$ is a positive but intensive quantity. Since this difference vanishes in thermodynamic limit, the relative stability between these two states is very delicate. To discuss the stability of the ground state $|F\rangle$, it is necessary to understand its structure, which turns out to be quite remarkable.

Super- and coherent fragmentation.—Because of Bose statistics, the N-body singlet of a spin-1 Bose gas is unique. A simple exercise shows that $\Theta^\dagger \equiv -2a_1^\dagger a_{-1}^\dagger + a_0^{\dagger 2}$ creates a singlet pair. The ground state $|F\rangle = |S;S\rangle$ is therefore

$$|S;S\rangle = \frac{1}{\sqrt{f(Q;S)}}\,a_1^{\dagger S}\Theta^{\dagger Q}|\text{vac}\rangle, \qquad Q = (N-S)/2, \quad (3)$$

where $f(Q;S)$ is the normalization constant which can be shown to be [8]

$$f(Q;S) = S!\,Q!\,2^Q\,\frac{(2Q+2S+1)!!}{(2S+1)!!}. \quad (4)$$

With the aid of Eq. (4), it is easy to show that the single particle density matrix of $|F\rangle$ is diagonal, $(\hat{\rho}^F)_{\alpha\beta} = \langle a_\beta^\dagger a_\alpha\rangle_F = N_\alpha \delta_{\alpha\beta}$, with

$$N_1 = \frac{N(S+1) + S(S+2)}{2S+3},$$
$$N_{-1} = \frac{(N-S)(S+1)}{2S+3}, \qquad N_0 = \frac{N-S}{2S+3}. \quad (5)$$

Before discussing the significance of Eq. (5), let us further investigate the two-body correlations of the ground state $|F\rangle$. Defining $\hat{N}_\alpha \equiv a_\alpha^\dagger a_\alpha$, $\alpha = 0, \pm 1$, we note that because of the identities $\hat{N}_{-1} = \hat{N}_1 - S$, and $\hat{N}_0 = N + S - 2\hat{N}_1$, all two-body correlations can be expressed in terms of $\langle \hat{N}_1^2\rangle$, or the density fluctuation $(\Delta\hat{N}_1)^2 \equiv \langle (a_1^\dagger a_1 - \langle a_1^\dagger a_1\rangle)^2\rangle$. For example, we have $(\Delta\hat{N}_{-1})^2 = (\Delta\hat{N}_0)^2/4 = (\Delta\hat{N}_1)^2$; $\langle \hat{N}_0\hat{N}_{-1}\rangle = \langle \hat{N}_1\rangle(N+3S) - 2\langle \hat{N}_1^2\rangle SN - S^2$, etc. Again using Eq. (4), it is straightforward (though lengthy) to show that

$$(\Delta\hat{N}_1)^2 = \left(\frac{N}{2S+3}\right)^2\left(\frac{S+1}{2S+5}\right) + \left(\frac{3N}{(2S+3)^2}\right)\left(\frac{S+1}{2S+5}\right) + \left(\frac{S+1}{2S+5}\right)\left(\frac{S^2-3S}{(2S+3)^2}\right). \quad (6)$$

Equations (5) and (6) together illustrate the general behavior of the ground state $|F\rangle$ as a function of its magnetization S/N. Near $S = 0$, the system has three macroscopic eigenvalues $N_1 = N_0 = N_{-1} = N/3$ [7] and enormous fluctuations $\Delta\hat{N}_\alpha \sim N$. As S increases, both N_0 and the fluctuations $\{\Delta\hat{N}_\alpha\}$ shrink rapidly. When S becomes macroscopic, N_0 and all $\Delta\hat{N}_\alpha$ become order 1 (which essentially is zero in the thermodynamic limit), whereas $N_{\pm 1} \to (N \pm S)/2$ remain macroscopic. Since the system has more than one macroscopic eigenvalue for all magnetization, the ground state $|F\rangle$ is a *fragmented* condensate [2]. Equation (6) also shows that the single particle density matrix itself is insufficient to specify the nature of the system, as different fragmented condensates can have completely different two-particle correlations [such as those reflected in $(\Delta\hat{N}_\alpha)^2$]. To distinguish different fragmented states, we call those with $(\Delta\hat{N}_\alpha)^2 \sim N^2$ "superfragmented," and those with $(\Delta\hat{N}_\alpha)^2 \sim 1$ "coherent" fragmented.

In the thermodynamic limit, the superfragmented regime is a singularity which occurs only at $S/N = 0$. All other states with $S/N \neq 0$ are coherent fragmented. On the other hand, quantum gases are mesoscopic systems. The bulk singularity therefore turns into a crossover region (as a function of S/N). Setting $\Delta\hat{N}_\alpha \sim \sqrt{N}$ as the crossover estimate from coherent fragmentation to superfragmentation, Eq. (6) implies that (for $N, S \gg 1$) superfragmentation emerges when $S/N < 1/\sqrt{8N}$. For $N = 1800$, it means $S/N < 0.016$. Thus, superfragmentation can be achieved only when S/N is very close to zero. On the other hand, coherent fragmented states are more generic and are easier to realize.

The exceedingly small fluctuations $\Delta\hat{N}_\alpha$ in the coherent-fragmented regime means that the eigenstates $|S;S\rangle$ with large S can be well approximated by the Fock state,

$$|S;S\rangle \to |N_1, N_{-1}, 0\rangle \equiv \frac{a_1^{\dagger N_1} a_{-1}^{\dagger N_{-1}}}{\sqrt{N_1!\,N_{-1}!}}|\text{vac}\rangle, \quad (7)$$

where $N_{\pm 1} = (N \pm S)/2$ and $|N_1, N_{-1}, N_0\rangle$ denotes a state with N_α bosons in spin state α, $\alpha = 0, \pm 1$. Within the Fock space, $\hat{H}_o$ in Eq. (1) becomes

$$\hat{H}_o = \frac{c_2}{2\Omega}(\hat{N}_1 - \hat{N}_{-1})^2 - \gamma B(\hat{N}_1 - \hat{N}_{-1}). \quad (8)$$

Equation (7) amounts to dropping the a_0 contribution in the singlet operator Θ in Eq. (3). The possibility of such a reduction for large S is a consequence of bosonic enhancement; i.e., the amplitude for adding a particle to a condensate with N bosons scales as $\sqrt{N+1}$, $a^\dagger|N\rangle = \sqrt{N+1}|N+1\rangle$. As a result, the coefficient of the term $(a_1^\dagger a_{-1}^\dagger)^{(N-S)/2}$ in the product $a_1^{\dagger S}(-2a_1^\dagger a_{-1}^\dagger + a_0^{\dagger 2})^{(N-S)/2}$ is most magnified because it contains mostly $\alpha = 1$ bosons, thus reducing Eq. (3) to Eq. (7).

It is useful to compare the Fock state Eq. (7) with the coherent state $|C\rangle$ [Eq. (2)]. Denoting $|\ell\rangle \equiv |N_1 + \ell, N_{-1} - \ell, 0\rangle$, we can write Eq. (2) as $|C\rangle = \sum_{\ell=-N_1}^{N_{-1}} (\frac{N_1^{N_1+\ell} N_{-1}^{N_{-1}-\ell}}{(N_1+\ell)!(N_{-1}-\ell)!} \frac{N!}{N^N})^{1/2}|\ell\rangle$. Using a Stirling formula, we see that the coherent state with spin S is a Gaussian sum of Fock states,

$$|C\rangle \cong (\pi\sigma^2)^{-1/4} \sum_{\ell=-N_1}^{N_{-1}} e^{-\ell^2/2\sigma^2} e^{\xi\ell}|\ell\rangle, \tag{9}$$

where $\sigma^2 = 2N_1 N_{-1}/N$, $\xi = (N_{-1}^{-1} - N_1^{-1})/4$, $N_{\pm 1} = (N \pm S)/2$ [9]. Within the space of $\mu = \pm 1$, the density matrices $\hat{\rho}^C = \langle a_\mu^\dagger a_\nu\rangle_C$ and $\hat{\rho}^F = \langle a_\mu^\dagger a_\nu\rangle_F$ of the coherent state and the Fock state are given by

$$\hat{\rho}^C_{\mu\nu} = \begin{pmatrix} N_1 & \sqrt{N_1 N_{-1}} \\ \sqrt{N_1 N_{-1}} & N_{-1} \end{pmatrix}, \qquad \hat{\rho}^F_{\mu\nu} = \begin{pmatrix} N_1 & 0 \\ 0 & N_{-1} \end{pmatrix}. \tag{10}$$

They differ by the large off diagonal elements $\sqrt{N_1 N_{-1}}$.

Reassembling fragmented states into single condensates.—The origin of fragmentation in the spin-1 case turns out to be strict angular momentum conservation, which implies the density matrix to be diagonal (hence describing a fragmented condensate) because the operators $a_\mu^\dagger a_\nu$ with $\mu \neq \nu$ change the total S_z of the system. To turn a fragmented condensate into a coherent one, it is necessary to have processes to flip the spin from 1 to -1. The natural candidate for such an interaction is a magnetic field gradient G', which exists in the MIT experiment [6]. To be concrete, we take $\mathbf{B}(\mathbf{x}) = B_o(\hat{\mathbf{z}} + G'[x\hat{\mathbf{x}} - z\hat{\mathbf{z}}])$. The condition of small field gradient is that $G'\Omega^{-1/3} \ll 1$. It is useful to choose the local field direction $\hat{\mathbf{B}}(\mathbf{r})$ as the spin quantization axis. This is done by performing a unitary transformation $\hat{U} = \prod_{i=1}^N e^{-i\boldsymbol{\theta}(\mathbf{x}_i)\cdot\mathbf{S}_i}$, where $\boldsymbol{\theta} = \hat{\mathbf{z}} \times \hat{\mathbf{B}} = G'x\hat{\mathbf{y}} + 0(G'^2)$. Under this transformation, $\hat{V}$ is invariant because it is a spin conserving contact interaction. On the other hand, the momentum $\mathbf{p}$ transforms as $\hat{U}^\dagger \mathbf{p}\hat{U} = \mathbf{p} + \hbar G' S_y \hat{\mathbf{x}} + 0(G'^3)$. This transforms $\hat{H}_o$ into $\tilde{H}_o \equiv (\hat{U}^\dagger \hat{H}\hat{U})_o = \hat{H}_o + \hat{H}_1$, where $\hat{H}_1 = \epsilon \sum_{i=1}^N (S_i^y)^2$ and $\epsilon \equiv \frac{\hbar^2 G'^2}{2M}$. Restricting to the Hilbert space of the Fock states [Eq. (7)], $\hat{H}_1$ becomes $\hat{H}_1 = -(\epsilon/2)(a_1^\dagger a_{-1} + a_{-1}^\dagger a_1) + \epsilon N/2$. The effective Hamiltonian $\tilde{H}_o$ is then

$$\tilde{H}_o = -\frac{\epsilon}{2}(a_1^\dagger a_{-1} + \mathrm{H.c.}) + \frac{c_2}{2\Omega}(\hat{N}_1 - \hat{N}_{-1})^2 - \gamma B(\hat{N}_1 - \hat{N}_{-1}). \tag{11}$$

The effect of $\hat{H}_1$ on the unperturbed ground state $|N_1, N_{-1}, 0\rangle$ is to generate the set $\{|\ell\rangle \equiv |N_1 + \ell, N_{-1} - \ell, 0\rangle\}$. Within this set, $\tilde{H}_o$ has the tight-binding form

$$\tilde{H}_o = \sum_{\ell=0,\pm1,\dots} \left[\frac{2c_2\ell^2}{\Omega}|\ell\rangle\langle\ell| - \frac{t_\ell}{2}(|\ell+1\rangle\langle\ell| + \mathrm{H.c.})\right], \tag{12}$$

where $t_\ell \equiv \epsilon\sqrt{(N_1 + \ell + 1)(N_{-1} - \ell)}$. Writing the eigenstates of Eq. (12) as $|\Psi\rangle = \sum_\ell \Psi_\ell|\ell\rangle$, the Schrödinger equation is $E\Psi_\ell = (2c_2/\Omega)\ell^2\Psi_\ell - (t_\ell\Psi_{\ell+1} + t_{\ell-1}\Psi_{\ell-1})/2$. Replacing $(t_\ell + t_{\ell-1})/2 \sim \epsilon\sqrt{N_1 N_{-1}}$, and $(t_\ell - t_{\ell-1})/2 \sim -\epsilon\sqrt{N_1 N_{-1}}\,\xi$, where $\xi = (N_{-1}^{-1} - N_1^{-1})/4$ as defined before, the Schrödinger equation then becomes

$$\left(\frac{\Omega E}{4c_2\eta^4} + 1\right)\Psi_\ell = -\frac{1}{2}\frac{d^2\Psi_\ell}{d\ell^2} + \xi\frac{d\Psi_\ell}{d\ell} + \frac{\ell^2}{2\eta^4}\Psi_\ell, \tag{13}$$

$$\eta^4 \equiv \epsilon\sqrt{N_1 N_{-1}}\,\Omega/(4c_2). \tag{14}$$

The normalized ground state of Eq. (13) and its density matrix $(\hat{\rho}_\Psi)_{\alpha\beta} = \langle\Psi|a_\alpha^\dagger a_\beta|\Psi\rangle$ $(\alpha, \beta = \pm 1)$ are [9]

$$|\Psi\rangle = (\pi\eta^2)^{-1/4} \sum_\ell e^{-\ell^2/2\eta^2} e^{\xi\ell}|\ell\rangle, \tag{15}$$

$$(\hat{\rho}_\Psi)_{\alpha\beta} = \begin{pmatrix} N_1 & \sqrt{N_1 N_{-1}}\,e^{-1/4\eta^2} \\ \sqrt{N_1 N_{-1}}\,e^{-1/4\eta^2} & N_{-1} \end{pmatrix}. \tag{16}$$

The eigenvalues of $\hat{\rho}_\Psi$ are

$$\lambda_\pm = \frac{1}{2}\left[N \pm \sqrt{N^2 e^{-1/2\eta^2} + S^2(1 - e^{-1/2\eta^2})}\right]. \tag{17}$$

For zero field gradient, $\eta \to 0$, $\lambda_\pm \to \frac{1}{2}(N \pm S)$. The condensate is fragmented. For large field gradients, $\eta \gg 1$, Eq. (15) reduces to Eq. (2), $\lambda_+ \to N$, and $\lambda_- \to 0$; i.e., the system becomes a single coherent condensate. Note that even for $\eta \sim 5$ the system is essentially a single condensate state. Using the expression of c_2 in Ref. [4], $c_2 = 4\pi\hbar^2\Delta a_{\rm sc}/M$, $\Delta a_{\rm sc} = (a_2 - a_0)/3$, we have, from Eq. (14), $\eta \equiv \eta_o[1 - (\frac{S}{N})]^{1/8}$, and $\eta_o = [\epsilon\Omega N/(4c_2)]^{1/4} = [\frac{(G'\Omega^{1/3})^2 N^{4/3}}{32\pi(\Delta a_{\rm sc} n^{1/3})}]^{1/4}$. Since $\eta_o \sim N^{1/3}$, arbitrarily small field gradients will change a fragmented state into a single condensate for a homogeneous Bose gas in the thermodynamic limit. However, trapped Bose gases with $N \lesssim 10^6$ are mesoscopic. In this limit, super- and coherent-fragmented states are no longer singularities in the phase space $(\frac{S}{N}, \epsilon)$. Instead, they occupy a finite region in the phase space which crosses over to single condensate states in a continuous manner. This implies the possibility of observing fragmented condensates in trapped gases.

Trapped spin-1 Bose gas.—When $\mathbf{B} \neq 0$, different spin components have different spatial extents. For cylindrical traps, we write $\psi_\mu(\mathbf{x}) = f_\mu(\mathbf{x})a_\mu + \phi_\mu(\mathbf{x})$, where f_μ's are normalized and cylindrically symmetric wave functions of the condensate with spin μ, and ϕ_μ is the noncondensate part of ψ_μ satisfying $\int f_\mu \phi_\mu = 0$. The exact forms of f_μ are determined by energy minimization. As in the homogeneous case, we write $\hat{H} = \hat{H}_o + \hat{H}_{\rm ex}$, where $\hat{H}_o$ contains only a_μ. Our first step is to note that, when the magnetization become macroscopic, the ground state of the trapped gas is accurately given by the Fock state $|F\rangle = |N_1, N_{-1}, 0\rangle$, with $N_{\pm 1} = (N \pm S)/2$, as in the homogeneous case [10]. The operator $\hat{\psi}_0$ then drops out from the problem. The functions f_1 and f_{-1} are determined by minimizing $E = \langle \hat{H}_o \rangle_F$. A repeat of our previous calculation shows that in the presence of a field gradient the effective Hamiltonian $\hat{H}'_o$ in a frame aligned with the external field is again given by Eq. (12), with $\Omega^{-1} \to \frac{1}{4}\int[(f_1^2 + f_{-1}^2)^2 + \frac{c_0}{c_2}(f_1^2 - f_{-1}^2)^2]$, and $\epsilon \to \epsilon \int f_1 f_{-1}$.

To estimate the field gradients G' and magnetization S/N needed for observing fragmented states, we note that the largest eigenvalue λ_+ of the density matrix decreases from N to $(N + S)/2$ as one moves from a coherent to a Fock regime. The Fock regime emerges approximately when $\lambda_+ < \lambda^* = \frac{1}{2}[N + \frac{1}{2}(N + S)]$. As seen from Eq. (14), η depends on S/N because $N_{\pm 1} = (N \pm S)/2$, and on G' through ϵ. The condition $\lambda_+ < \lambda^*$ for the appearance of a Fock regime can be cast as a condition on G' and S/N. More precisely, Eq. (14) implies $\eta^4 = \frac{N\Omega G'^2}{32\pi\Delta a_{\rm sc}}[1 - (\frac{S}{N})^2]^{1/2}$, where we have used $c_2 = 4\pi\hbar^2\Delta a_{\rm sc}/M$. The condition $\lambda_+ < \lambda^*$ then implies $G' < (\frac{8\pi\Delta a_{\rm sc}}{\sqrt{N_1 N_{-1}}\Omega[\int f_1 f_{-1}]})^{1/2}/(\ln\frac{4(N+S)}{N+3S})$. For a ^{23}Na gas ($\Delta a_{\rm sc} \sim 10^{-8}$ cm) with $N = 10^6$, $S/N = 0.2$, we have $\lambda^*/N = 0.8$. Since Ω is roughly the volume of the system, if we take for $\Omega \int f_1 f_{-1} \sim 10^{-12}$ cm^3, we have $\lambda < \lambda^*$ when $G' < 0.46$ cm^{-1}.

Distinguishing fragmented states from single condensate states.—The difference between a Fock state and a coherent state of a scalar condensate in a double well is a subject of recent discussions [11]. While it was shown that these two states cannot be distinguished in typical interference experiments [11], it is generally believed that Josephson tunneling can exist in a coherent state but not in a fragmented state [12]. Although experiments on Josephson tunneling have not been performed, there appears no intrinsic reason preventing its existence or observability. The Hamiltonian $\tilde{H}_0$ [Eq. (11)] of a spin-1 Bose gas is identical to that of a scalar Bose gas in a double well with a bias potential, where "1" and "-1" now label the two wells; $\epsilon/2$, $c_2/2\Omega$, and γB play the role of tunneling matrix element t, the repulsive interaction U of the scalar Bose gas, and the bias potential V. In the case of a double well, if the ground state of the system is a coherent state, a new state will result when a phase difference ϕ is introduced to the condensates on the different sides of the barrier, i.e., $(a_1^\dagger + a_2^\dagger)^N|\text{vac}\rangle \to (e^{-i\phi/2}a_1^\dagger + e^{i\phi/2}a_2^\dagger)^N|\text{vac}\rangle$; and subsequent Josephson oscillation will occur with a current which is in general macroscopic (of order $N\epsilon$) as a consequence of the identity $\frac{d\hat{N}_1}{dt} = \frac{\epsilon}{2i}(a_1^\dagger a_{-1} - a_{-1}^\dagger a_1) \equiv \hat{I}$. In contrast, as shown in Ref. [12], Josephson oscillation will not occur in the Fock regime.

For scalar Bose gas in double wells, the phase difference ϕ can be generated by using the phase-imprinting method. For spin-1 Bose gas, the phase difference can be generated by applying a short but large uniform magnetic pulse, i.e. changing B to $B + \Delta B$ over a short time interval τ such that $\gamma\Delta B \gg \sqrt{UN\epsilon}$ (i.e., oscillation frequency of the junction), and $\tau\gamma\Delta B \sim 1$. This will change the coherent ground state as mentioned before with $\phi = 2\gamma\Delta B\tau$.

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[8] To derive Eq. (4), we define $A_x = (-a_1 + a_{-1})/\sqrt{2}$, $A_y = -i(a_1 + a_{-1})/\sqrt{2}$, $A_z = a_0$. We then have $\mathbf{A}^2 = \Theta$. Writing $|Q,S\} = a_1^{\dagger S}\Theta^{\dagger Q}|\text{vac}\rangle$ as $|Q,S\} = [T_S^Q(\mathbf{x})e^{\mathbf{A}^\dagger\cdot\mathbf{x}}|\text{vac}\rangle]_o$, where $T_S^Q(\mathbf{x}) \equiv (-2^{1/2})^S(\partial_x + i\partial_y)^S(\nabla^2)^Q$, and $(\ldots)_o$ means $\mathbf{x} = 0$, and noting that $\langle\text{vac}|e^{\mathbf{A}\cdot\mathbf{x}}e^{\mathbf{A}^\dagger\cdot\mathbf{x}'}|\text{vac}\rangle = e^{\mathbf{x}\cdot\mathbf{x}'}$, we have $f(Q,S) = \{Q,S\,|\,Q,S\} = [T_S^{*Q}(\mathbf{x})T_S^Q(\mathbf{x}')e^{\mathbf{x}\cdot\mathbf{x}'}]_o = [T_S^{*Q}(\mathbf{x})T_S^Q(\mathbf{x}')(\mathbf{x}\cdot\mathbf{x}')^{2Q+S}]_o/(2Q+S)!$, which gives Eq. (4) after differentiation.

[9] We have ignored a factor $e^{\xi^2\eta^2/2}$ in the normalization of $|\Psi\rangle$ because $\xi^2\eta^2 \ll 1$.

[10] This can be shown by separating out the terms in $\hat{H}$ which contains only number operators $\{\hat{N}_\alpha\}$ (denoted as $\hat{H}_A$) and those that mix a_0 with $a_{\pm 1}$ (denoted as $\hat{H}_B$). One can then see that $\hat{H}_A$ is minimized when $N_0 = 0$ and that the contribution of $\hat{H}_B$ contributes only to an $1/N$ correction to the energy and can hence be ignored.

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