

# Entanglement Purification of Gaussian Continuous Variable Quantum States

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We describe an entanglement purification protocol to generate maximally entangled states with high efficiencies from two-mode squeezed states or from mixed Gaussian continuous entangled states. The protocol relies on a local quantum nondemolition measurement of the total excitation number of several continuous variable entangled pairs. We propose an optical scheme to do this kind of measurement using cavity enhanced cross-Kerr interactions.

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Quantum communication, such as quantum key distribution and quantum teleportation, is hampered by the difficulty to generate maximally entangled states between distant nodes [1]. Because of loss and decoherence, in reality we can generate only partially entangled states between distant sides [2]. Entanglement purification techniques are needed to concentrate maximally entangled states from partially entangled states [3,4]. For qubit systems, efficient entanglement purification protocols have been found [3–5]. But none of these purification schemes have been realized experimentally due to the great difficulty of performing repeated collective operations in realistic quantum communication systems. Thus, it is of interest to consider purification of continuous variable entanglement. The non-local Gaussian continuous variable entangled states (i.e., states whose Wigner functions are Gaussians) can be easily generated by transmitting two-mode squeezed light, and this kind of entanglement has been demonstrated in the recent experiment of continuous variable teleportation [6]. As the first choice for performing continuous entanglement purification, one would consider direct extensions of the purification schemes for qubit systems. But until now, in these extensions, no entanglement increase has been found for Gaussian continuous entangled states [7]. Thus, the discussion should be extended to a larger class of operations to purify continuous entangled states. Braunstein *et al.* [8] have proposed a simple error correction scheme for continuous variables. However, it is not clear whether it can be used for purification. In [9] a protocol to increase the entanglement for the special case of pure two-mode squeezed states has been proposed, which is based on conditional photon number subtraction; the efficiency, however, seems to be an obstacle for its practical realization.

In this paper, we present an entanglement purification scheme with the following properties: (i) For pure states it reaches the maximal allowed efficiency in the asymptotic limit (when the number of pairs of modes goes to infinity). (ii) It can be readily extended to distill maximally entangled states from a relevant class of mixed

Gaussian states which result from losses in the light transmission. Furthermore, we propose and analyze a scheme to implement this protocol experimentally using high finesse cavities and cross-Kerr nonlinearities. Our purification protocol generates maximally entangled states in finite dimensional Hilbert spaces. The entanglement in the continuous partially entangled state is transformed to the maximally entangled state with a high efficiency. We begin the paper by describing the entanglement purification protocol for pure two-mode squeezed states, then extend the protocol to include mixed Gaussian continuous states, and last describe the physical implementation of the purification protocol.

First, assume that we have generated  $m$  entangled pairs  $A_i, B_i$  ( $i = 1, 2, \dots, m$ ) between two distant sides A and B. Each pair of modes  $A_i, B_i$  are prepared in the two mode squeezed state  $|\Psi\rangle_{A_i B_i}$ , which in the number basis has the form

$$|\Psi\rangle_{A_i B_i} = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n\rangle_{A_i} |n\rangle_{B_i}, \quad (1)$$

where  $\lambda = \tanh(r)$ , and  $r$  is the squeezing parameter [10]. For and only for a pure state, the entanglement is uniquely quantified by the von Neumann entropy of the reduced density operator of its one-component. The entanglement of the state (1) is thus given by  $E(|\Psi\rangle_{A_i B_i}) = \cosh^2(r) \log[\cosh^2(r)] - \sinh^2(r) \log[\sinh^2(r)]$ . The joint state  $|\Psi\rangle_{(A_i B_i)}$  of the  $m$  entangled pairs is simply the product of all the  $|\Psi\rangle_{A_i B_i}$ , which can be rewritten as

$$|\Psi\rangle_{(A_i B_i)} = (1 - \lambda^2)^{m/2} \sum_{j=0}^{\infty} \lambda^j \sqrt{f_j^{(m)}} |j\rangle_{(A_i B_i)}, \quad (2)$$

where  $(A_i B_i)$  is an abbreviation of the symbol  $A_1, B_1, A_2, B_2, \dots$ , and  $A_m, B_m$ , and the normalized state  $|j\rangle_{(A_i B_i)}$  is defined as

$$|j\rangle_{(A_i B_i)} = \frac{1}{\sqrt{f_j^{(m)}}} \sum_{i_1, i_2, \dots, i_m}^{i_1 + i_2 + \dots + i_m = j} |i_1, i_2, \dots, i_m\rangle_{(A_i)} \otimes |i_1, i_2, \dots, i_m\rangle_{(B_i)}. \quad (3)$$

The function  $f_j^{(m)}$  in Eqs. (2) and (3) is given by  $f_j^{(m)} = \frac{(j+m-1)!}{j!(m-1)!}$ . To concentrate entanglement of these  $m$  entangled pairs, we perform a quantum nondemolition (QND) measurement of the total excitation number  $n_{A_1} + n_{A_2} + \dots + n_{A_m}$  on the A side (we will describe later how to implement this measurement experimentally). The QND measurement projects the state  $|\Psi\rangle_{(A_i B_i)}$  onto a two-party maximally entangled state  $|j\rangle_{(A_i B_i)}$  with probability  $p_j = (1 - \lambda^2)^m \lambda^{2j} f_j^{(m)}$ . The entanglement of the outcome state  $|j\rangle_{(A_i B_i)}$  is given by  $E(|j\rangle_{(A_i B_i)}) = \log(f_j^{(m)})$ . The quantity  $\Gamma_j = E(|j\rangle_{(A_i B_i)})/E(|\Psi\rangle_{A_i B_i})$  defines the entanglement increase ratio, and, if  $\Gamma_j > 1$ , we get a more entangled state. Even with a small number  $m$ , the probability of getting a more entangled state is quite high. It can be easily proven that, if  $m$  goes to infinity, with unit probability we would get a maximally entangled state with entanglement  $mE(|\Psi\rangle_{A_i B_i})$ . This ensures that this method is optimal in this limit, analogous to the purification protocol presented in [3] for the qubit case. For any finite number of entangled pairs, the present purification protocol is more efficient than that in [3], since it takes advantage of the special relations between the coefficients in the two-mode squeezed state.

An interesting feature of this entanglement purification protocol is that for any measurement outcome  $j \neq 0$  we always get a useful maximally entangled state in some finite Hilbert space, though the entanglement of the outcome state  $|j\rangle_{(A_i B_i)}$  does not necessarily exceed that of the original state  $|\Psi\rangle_{A_i B_i}$  if  $j$  is small. It is also interesting to note that a small alternation of this scheme provides a useful method for preparing GHZ-like (Greenberger-Horne-Zeilinger) states in high dimensional Hilbert spaces [11]. The key point is that the modes  $B_i$  need not be at the same side in the protocol. Assume we have two entangled pairs  $B, A_1$  and  $A_2, C$  distributed at three sides B, A, C, with each pair being prepared in the state (1). Then a local QND measurement of the modes  $A_1, A_2$  at the A side with the outcome  $j \neq 0$  generates a three-party GHZ state in the  $(j+1)$ -dimensional Hilbert space. Obviously, if we have  $m$  entangled pairs, we can generate a  $(m+1)$ -party GHZ state using this method.

In reality, the light transmission will be unavoidably subjected to loss, and then we will not start from an ideal two-mode squeezed state, but instead from a mixed state described by the following master equation:

$$\dot{\rho} = -i(H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger) + \sum_{i=1}^m (\eta_A a_{A_i} \rho a_{A_i}^\dagger + \eta_B a_{B_i} \rho a_{B_i}^\dagger), \quad (4)$$

where  $\rho$  is the density operator of the  $m$  entangled pairs with  $\rho(0) = |\Psi\rangle_{(A_i B_i)}\langle\Psi|$ , the ideal two-mode squeezed state, and the effective Hamiltonian,

$$H_{\text{eff}} = -i \sum_{i=1}^m \left( \frac{\eta_A}{2} a_{A_i}^\dagger a_{A_i} + \frac{\eta_B}{2} a_{B_i}^\dagger a_{B_i} \right). \quad (5)$$

In Eqs. (4) and (5),  $a_{\alpha_i}$  denotes the annihilation operator of the mode  $\alpha_i$  ( $\alpha = A$  or  $B$ ), and we have assumed that the damping rates  $\eta_A$  and  $\eta_B$  are the same for all the  $m$  entangled pairs based on symmetry considerations, but  $\eta_A$  and  $\eta_B$  may be different to each other.

In many practical cases, it is reasonable to assume that the light transmission noise is small. Let  $\tau$  denote the transmission time, then  $\eta_A \tau$  and  $\eta_B \tau$  are small factors. In the language of quantum trajectories [10], to the first order of  $\eta_A \tau$  and  $\eta_B \tau$ , the final state of the  $m$  entangled pairs is either  $|\Psi^{(0)}\rangle_{(A_i B_i)} \propto e^{-iH_{\text{eff}}\tau} |\Psi\rangle_{(A_i B_i)}$ , with no quantum jumps occurred, or  $|\Psi^{(\alpha_i)}\rangle_{(A_i B_i)} \propto \sqrt{\eta_{\alpha} \tau} a_{\alpha_i} |\Psi\rangle_{(A_i B_i)}$ , with a jump occurred in the  $\alpha_i$  channel ( $\alpha = A, B$  and  $i = 1, 2, \dots, m$ ). The final density operator is a mixture of all these possible states. To purify entanglement from the mixed state, we perform QND measurements of the total excitation number on both sides A and B, and the measurement results are denoted by  $j_A$  and  $j_B$ , respectively. We then compare  $j_A$  and  $j_B$  through classical communication, and keep the outcome state if and only if  $j_A = j_B$ . Let  $P_A^{(j)}$  and  $P_B^{(j)}$  denote the projections onto the eigenspaces of the corresponding total number operators  $\sum_{i=1}^m a_{A_i}^\dagger a_{A_i}$  and  $\sum_{i=1}^m a_{B_i}^\dagger a_{B_i}$  with eigenvalue  $j$ , respectively. It is easy to show that

$$\begin{aligned} P_A^{(j)} P_B^{(j)} |\Psi^{(0)}\rangle_{(A_i B_i)} &= |j\rangle_{(A_i B_i)}, \\ P_A^{(j)} P_B^{(j)} |\Psi^{(\alpha_i)}\rangle_{(A_i B_i)} &= 0. \end{aligned} \quad (6)$$

So, if  $j_A = j_B = j$ , the outcome state is the maximally entangled state  $|j\rangle_{(A_i B_i)}$  with entanglement  $\log(f_j^{(m)})$ . The probability to get the state  $|j\rangle_{(A_i B_i)}$  is now given by  $p_j' = (1 - \lambda^2)^m \lambda^{2j} f_j^{(m)} e^{-(\eta_A + \eta_B)\tau j}$ . It should be noted that the projection operators  $P_A^{(j)} P_B^{(j)}$  cannot eliminate the states obtained from the initial state  $|\Psi\rangle_{(A_i B_i)}$  by a quantum jump on each side A and B. The total probability for occurrence of these kinds of quantum jumps is proportional to  $m^2 \bar{n}^2 \eta_A \eta_B \tau^2$ . So the condition for small transmission noise requires  $m^2 \bar{n}^2 \eta_A \eta_B \tau^2 \ll 1$ , where  $\bar{n} = \sinh^2(r)$  is the mean photon for a single mode.

In the purification for mixed entanglement, we need classical communication (CC) to confirm that the measurement outcomes of the two sides are the same, and during this CC we implicitly assume that the storage noise for the modes is negligible. In fact, that the storage noise is much smaller than the transmission noise is a common assumption taken in all the entanglement purification schemes which need the help of repeated CCs [4,5]. If we also make this assumption for continuous variable systems, there exists another simple configuration for the purification protocol to work. We put the generation setup for two-mode squeezed states on the A side. After state generation, we keep the modes  $A_i$  on side A with a very small storage loss rate  $\eta_A$ , and at the same time the modes  $B_i$  are transmitted to the distant side B with a loss rate

$\eta_B \gg \eta_A$ . We call this a configuration with an asymmetric transmission noise. In this configuration, the purification protocol is exactly the same as that described in the above paragraph. We note that the component in the final mixed density operator which is kept by the projection  $P_A^{(j)} P_B^{(j)}$  should be subjected to the same times of quantum jumps on each side A and B. We want this component to be a maximally entangled state. This requires that the total probability for sides A and B to subject to the same nonzero times of quantum jumps should be very small. This total probability is always smaller than  $\bar{n} \eta_A \tau$ , despite how large the damping rate  $\eta_B$  is. So the working condition of the purification protocol in the asymmetric transmission noise configuration is given by  $\bar{n} \eta_A \tau \ll 1$ . The loss rate  $\eta_B$  can be large. The probability to get the maximally entangled state  $|j\rangle_{(A,B)}$  is still given by  $p_j' = (1 - \lambda^2)^m \lambda^{2j} f_j^{(m)} e^{-(\eta_A + \eta_B)\tau j}$ .

For continuous variable systems, the assumption of storage with a very small loss rate is typically unrealistic. If this is the case, then we can use the following simple method to circumvent the storage problem. Note that the purpose to distill maximally entangled states is to directly apply them in some quantum communication protocols, such as in quantum cryptography or in quantum teleportation. So we can modify the above purification protocol by the following procedure: right after the state generation, we take a QND measurement of the total excitation number on side A and get a measurement result  $j_A$ . Then we do not store the outcome state on side A, but immediately use it (e.g., perform the corresponding measurement as required by a quantum cryptography protocol [12]). During this process, the modes  $B_i$  are being sent to the distant side B and, when they arrive, we take another QND measurement of the total excitation number of the modes  $B_i$  and get an outcome  $j_B$ . The resulting state on side B can be directly used (for quantum cryptography, for instance) if  $j_A = j_B$ , and discarded otherwise. By this method, we formally get maximally entangled states through posterior confirmation, and at the same time we need not store the modes on both sides.

To experimentally implement the above purification scheme, we need first generate Gaussian continuous entangled states between two distant sides, and then perform a local QND measurement of the total excitation number of several entangled pairs. Here we propose a promising experimental scheme, which uses a high finesse optical cavity to carry continuous entangled states and cavity enhanced cross-Kerr interactions to realize the local QND measurement. It is possible to generate Gaussian continuous entangled states between two distant cavities [13]. We can transmit and then couple the two output lights of the nondegenerate optical parametric amplifier to distant high finesse cavities. The steady state of the cavities is just a Gaussian continuous entangled state described by the solution of Eq. (4) after taking into account the propagation loss [14]. The difficult part is to perform a QND

measurement of the total photon number contained in several local cavities. We use the setup depicted in Fig. 1 to attain this goal. (For convenience, we use the two-cavity measurement as an example to illustrate the method. Extension of the measurement method to multicavity cases is straightforward.)

The measurement model depicted in Fig. 1 is an example of the cascaded quantum system [10]. The incident light  $b_{i1}$  can be expressed as  $b_{i1} = b_{i1}' + g\sqrt{\gamma}$ , where  $g\sqrt{\gamma}$  ( $g$  is a large dimensionless factor) is a constant driving field, and  $b_{i1}'$  is the standard vacuum white noise, satisfying  $\langle b_{i1}'^\dagger(t)b_{i1}'(t') \rangle = 0$  and  $\langle b_{i1}'(t)b_{i1}'^\dagger(t') \rangle = \delta(t - t')$ . The Hamiltonian for the Kerr medium is assumed to be  $H_i = \hbar \chi n_i b_i^\dagger b_i$  ( $i = 1$  or  $2$ ), where  $b_i$  is the annihilation operator for the ring cavity mode, and  $\chi$  is the cross-phase modulation coefficient. The self-phase modulation can be made much smaller than the cross-phase modulation with some resonance conditions for the Kerr medium, and thus is negligible [15,16]. In the frame rotating at the optical frequencies, the Langevin equations describing the dynamics in the two ring cavities have the form

$$\begin{aligned} \dot{b}_1 &= -i\chi n_1 b_1 - \frac{\gamma}{2} b_1 - \sqrt{\gamma} b_{i1}' - g\gamma, \\ \dot{b}_2 &= -i\chi n_2 b_2 - \frac{\gamma}{2} b_2 - \sqrt{\gamma} b_{i2}, \end{aligned} \quad (7)$$

with the boundary conditions (see Fig. 1)  $b_{i2} = b_{o1} = b_{i1}' + g\sqrt{\gamma} + \sqrt{\gamma} b_1$  and  $b_{o2} = b_{i2} + \sqrt{\gamma} b_2$ . In the realistic case  $\gamma \gg \chi \langle n_i \rangle$  ( $i = 1, 2$ ), we can adiabatically eliminate the cavity modes  $b_i$ , and express the final output  $b_{o2}$  of the second ring cavity as an operator function of the observable  $n_1 + n_2$ . The experimentally measured quantity is the integration of the homodyne photon current over the measurement time  $T$ . Choosing the phase of

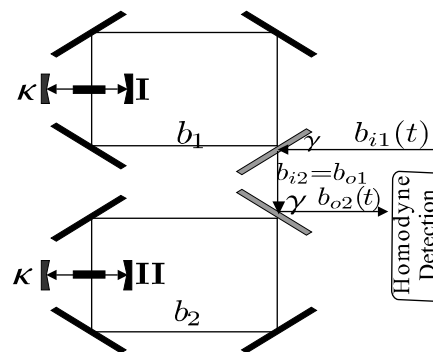


FIG. 1. Schematic experimental setup to measure the total photon number  $n_1 + n_2$  contained in the cavities I and II. The cavities I and II, each with a small damping rate  $\kappa$  and with a cross-Kerr medium inside, are put, respectively, in a bigger ring cavity. The ring cavities with the damping rate  $\gamma$  are used to enhance the cross-Kerr interactions. A strong continuous coherent driving light  $b_{i1}(t)$  is incident on the first ring cavity, whose output  $b_{o1}$  is directed to the second ring cavity. The output  $b_{o2}(t)$  of the second ring cavity is continuously observed through a homodyne detection.

the driving field so that  $g = i|g|$ , the measured observable corresponds to the operator

$$X_T = \frac{1}{T} \int_0^T \frac{1}{\sqrt{2}} [b_{o2}(t) + b_{o2}^\dagger(t)] dt \\ \approx \frac{4\sqrt{2}|g|\chi}{\sqrt{\gamma}} (n_1 + n_2) + \frac{1}{\sqrt{T}} X_T^{(b)}, \quad (8)$$

where  $X_T^{(b)} = \frac{1}{\sqrt{2}}(b_T + b_T^\dagger)$ , and  $b_T$ , satisfying  $[b_T, b_T^\dagger] = 1$ , is defined by  $b_T = 1/\sqrt{T} \int_0^T b_{i1}(t) dt$ . Equation (8) assumes  $\gamma \gg \chi \langle n_i \rangle$  and  $e^{-\gamma T} \ll 1$ . There are two different contributions in Eq. (8). The first term represents the signal, which is proportional to  $n_1 + n_2$ , and the second term is the vacuum noise. The distinguishability of this measurement is given by  $\delta n = \sqrt{\gamma}/(8|g|\chi\sqrt{T})$ . If  $\delta n < 1$ , i.e., if the measuring time  $T > \frac{\gamma}{64|g|^2\chi^2}$ , we effectively perform a measurement of  $n_1 + n_2$ ; and, if  $T$  is also smaller than  $\frac{1}{\kappa \langle n_i \rangle}$ , the photon loss in the cavities I and II during the measurement is negligible. So the setup gives an effective QND measurement of the total photon number operator  $n_1 + n_2$  under the condition

$$\frac{\gamma}{64|g|^2\chi^2} < T < \frac{1}{\kappa \langle n_i \rangle}. \quad (9)$$

This condition seems to be feasible with the present technology. For example, if we assume the cross-Kerr interaction is provided by the resonantly enhanced Kerr nonlinearity as considered and demonstrated in [15,16], the Kerr coefficient  $\chi/2\pi \sim 0.1$  MHz would be obtainable [17]. We can choose the decay rates  $\kappa/2\pi \sim 4$  MHz and  $\gamma/2\pi \sim 100$  MHz, and let the dimensionless factor  $g \sim 100$  (for a cavity with cross area  $S \sim 0.5 \times 10^{-4}$  cm<sup>2</sup>,  $g \sim 100$  corresponds to a coherent driving light with intensity about 40 mW cm<sup>-2</sup>). The mean photon number  $\langle n_1 \rangle = \langle n_2 \rangle = \sinh^2(r) \sim 1.4$  for a practical squeezing parameter  $r \sim 1.0$ . With the above parameters, Eq. (9) can be easily satisfied if we choose the measuring time  $T \sim 8$  ns. More favorable values for the parameters are certainly possible.

To bring the above proposal into a real experiment, there are several imperfect effects which should be considered. These imperfections include phase instability of the driving field, imbalance between the two ring cavities, light absorption of the Kerr media and the mirrors, self-phase

modulation effects, light transmission loss between the ring cavities, and inefficiency of the detectors. To realize a QND measurement, the imperfections should be small enough. We have deduced quantitative requirements for all the imperfections listed above [18]. With the parameters given in the above paragraph, all these requirements can be met experimentally.

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