Continuous Topological Phase Transitions between Clean Quantum Hall States

Xiao-Gang Wen
Department of Physics and Center for Materials Science and Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 27 August 1999)

Continuous transitions between states and the *same* symmetry but different topological orders are studied. Clean quantum Hall (QH) liquids with neutral nonbosonic quasiparticles are shown to have such transitions under the right conditions. For clean bilayer (mmn) states, a continuous transition to other QH states (including non-Abelian states) can be driven by increasing interlayer repulsion/tunneling. The effective theories describing the critical points at some transitions are obtained.

PACS numbers: 73.40.Hm, 73.20.Dx

Matters have several different states, such as solid, gas, superfluid, etc., under different conditions. According to Landau's theory, all of those states of matters are characterized by their symmetries. Quantum Hall (QH) liquids discovered in 1982 [1,2] in 2D electron gas form a new state of matter which contains a completely different type of order (called topological order [3]). The topological order is new since it is related to chiral operator product algebra instead of symmetries [3].

One way to gain a better understanding of the states of matter is to study *continuous* transition between different states. For the states characterized by symmetries, continuous transitions between them are characterized by a change of symmetry. The transition point is described by a critical theory which demonstrates various scaling properties. Similarly, to gain a better understanding of the topological orders, we can also study continuous transitions between different QH liquids. Since the topological orders are not characterized by symmetries, some fundamental questions naturally arise: (1) Do continuous phase transitions exist between different QH liquids? (2) Are the transition points described by critical theories with scaling properties?

In the presence of disorders, the transitions between quantum Hall liquids are believed to be continuous and are described by critical points. But in this paper, we would like to avoid the complication of disorders and would like to concentrate on a nonrandom system. It was pointed out in Ref. [4] that a continuous transition between two QH liquids of difference filling fractions can happen if we turn on a proper periodic potential. This allows us to study continuous phase transitions between QH liquids without introducing disorders. The transitions studied in Refs. [5,6] for paired states are other examples of continuous transitions (where a periodic potential is not needed). In this paper we apply the results from Ref. [4] to study the transitions between two QH liquids of the same filling fraction. We see, in this case, that the transitions can sometimes be continuous even without periodic potentials (and disorders). Such continuous transitions between clean QH states can actually appear in bilayer QH states. We will study some simple bilayer QH states and determine under which conditions we are likely to observe the continuous phase transition. The transitions between the paired states [5,6] are special cases of the transitions studied in this paper. Since the continuous transitions studied in this paper are transitions between two QH states with the *same* symmetry, they are fundamentally different from the usual continuous transitions which change the symmetry in the states. We call such transitions continuous topological phase transitions.

We start with an Abelian QH state described by $(K, \boldsymbol{q})$ with a Chern-Simons (CS) effective theory [7]

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} a_{I\mu}\partial_\nu a_{J\lambda}\varepsilon^{\mu\nu\lambda} - \frac{e}{2\pi} q_I A_\mu \partial_\nu a_{I\lambda}\varepsilon^{\mu\nu\lambda}, \tag{1}$$

where K is a symmetric integer matrix and $\boldsymbol{q}$ is an integer vector. The quasiparticles are labeled by integer vectors $\boldsymbol{l}$. The charge and the statistics of such a quasiparticle are given by

$$\theta_l = \pi \boldsymbol{l}^T K^{-1}\boldsymbol{l}, \qquad Q_l = -e\boldsymbol{l}^T K^{-1}\boldsymbol{q}. \tag{2}$$

Now assume that we have a gas of quasiparticles labeled by $\boldsymbol{l}$ on top of the $(K, \boldsymbol{q})$ state. The filling fraction of the quasiparticle gas is $\nu_q = \frac{n_q hc}{Q_l B}$, where n_q is the quasiparticle density. If $\nu_q = \frac{1}{n_e - \frac{\theta_l}{\pi}}$ for an even integer n_e, then the quasiparticles can form a Laughlin state, and we obtain a new Abelian QH liquid labeled by (see Blok and Wen in Ref. [7])

$$K' = \begin{pmatrix} K & -\boldsymbol{l} \\ -\boldsymbol{l}^T & n_e \end{pmatrix}, \qquad \boldsymbol{q}' = \begin{pmatrix} \boldsymbol{q} \\ 0 \end{pmatrix}. \tag{3}$$

Now let us consider a transition induced by condensation of *charge neutral* quasiparticles labeled by $\boldsymbol{l}$. We assume that the quasiparticles labeled by $\boldsymbol{l}$ and $-\boldsymbol{l}$ (the antiquasiparticles) have the lowest energy gap (so that they control the transition). The low energy effective theory for the quasiparticles and the antiquasiparticles has a form

$$\mathcal{L} = |(\partial_0 + ia_0)\phi|^2 - v^2|(\partial_i + ia_i)\phi|^2 - m^2|\phi|^2 - g|\phi|^4 - \frac{\pi}{\theta_l}\frac{1}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}. \tag{4}$$

Binding additional n_e flux quanta ($n_e =$ even) to the bosonic field ϕ gives us composite boson field $\tilde{\phi}$. The effective theory can also be written in terms of $\tilde{\phi}$,

 0031-9007/00/84(17)/3950(4)$15.00

$$\mathcal{L} = |(\partial_0 + ia_0 + ib_0)\tilde{\phi}|^2 - v^2|(\partial_i + ia_i + ib_i)\tilde{\phi}|^2 - m^2|\tilde{\phi}|^2 - g|\tilde{\phi}|^4 - \frac{\pi}{\theta_l}\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} + \frac{1}{n_e}\frac{1}{4\pi}b_\mu\partial_\nu b_\lambda\epsilon^{\mu\nu\lambda}. \tag{5}$$

Here $2m$ is the energy gap for creating a quasiparticle-antiquasiparticle pair.

Near the transition, two parameters m^2 and a_0 are important. (We can always assume $b_0 = 0$ without losing generality.) The (mean field) phase diagram is sketched in Fig. 1.

The transition at $a_0 = m^2 = 0$ is the transition from the $\langle\tilde{\phi}\rangle = 0$ phase [the $(K, \boldsymbol{q})$ phase] to the $\langle\tilde{\phi}\rangle \neq 0$ phase [the $(K', \boldsymbol{q}')$ phase]. Both $(K, \boldsymbol{q})$ and $(K', \boldsymbol{q}')$ phases have finite energy gap, while gapless neutral excitations appear at the transition point. The effective theory [Eq. (5)] for the transition is identical to the one studied in Ref. [4]. It was shown that the transition is continuous at least in a large N limit [4]. The transition point is a critical point with scaling properties. Some critical exponents at the transition point ($a_0 = m^2 = 0$) were calculated in the large N limit.

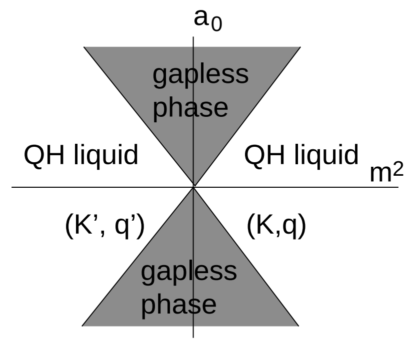

FIG. 1. The phase diagram near a critical point.

We see that a continuous transition between the two QH liquids $(K, \boldsymbol{q})$ and $(K', \boldsymbol{q}')$ can happen even without the lattice if $Q_l = 0$. *A QH state with neutral quasiparticles can have a continuous phase transition to another QH state even in the clean limit.* One can show that the two QH states always have the same filling fraction.

We can also bind an odd number n_o flux quanta to ϕ and use an effective composite fermion theory to describe the transition

$$\mathcal{L} = \psi^\dagger(\partial_\mu + ia_\mu + ib_\mu)\tau^\mu\psi - m\psi^\dagger\tau^3\psi + \Psi^\dagger(\partial_\mu + ia_\mu + ib_\mu)\tau^\mu\Psi - M\Psi^\dagger\tau^3\Psi - \frac{\pi}{\theta_l}\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} + \frac{1}{n_o}\frac{1}{4\pi}b_\mu\partial_\nu b_\lambda\epsilon^{\mu\nu\lambda}, \tag{6}$$

where $\tau^0 = 1$ and $\tau^i|_{i=1,2,3}$ are the Pauli matrices and ψ and Ψ are two-component fermion fields. M is a large number and Ψ is the regularization field. Integrating out ψ and Ψ generates a CS term $-\frac{\mathrm{sgn}(M)\theta(mM)}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}$. The (mean field) phase diagram for composite fermion effective theory is identical to that of composite boson (Fig. 1), if we replace m^2 by $-mM$. The critical properties of Eq. (6) at $a_0 = m = 0$ are studied in Ref. [8]. The transition are shown to be continuous in the limit $\theta_l \to \pi$ and $n_o = 0$. Both Eqs. (5) and (6) describe the same set of transitions, labeled by n_e or n_o. The two sets of labels are related by $n_e = n_o - \mathrm{sgn}(M)$.

In general, after transition, the new QH liquid described by $(K', \boldsymbol{q}')$ contains $\dim(K) + 1$ edge branches, since $\dim(K') = \dim(K) + 1$. The quantum number of the quasiparticles is given by Eq. (2). However, when $(K', \boldsymbol{q}')$ contains neutral null vectors [9,10], the $(K', \boldsymbol{q}')$ QH liquid is really a QH liquid described by a reduced $(\tilde{K}, \tilde{\boldsymbol{q}})$ with dimension $\dim(K) - 1$ [9,10]. A calculation of $(\tilde{K}, \tilde{\boldsymbol{q}})$ from $(K', \boldsymbol{q}')$ was outlined in Ref. [9]. We would like to point out that, if $n_e = 0$ in Eq. (3), $(K', \boldsymbol{q}')$ always has at least one neutral null vector $\boldsymbol{l}_{\mathrm{null}}^T = (\boldsymbol{l}^T, 0)$, and the transition reduces the number of edge branches.

Now let us study a concrete bilayer $(nn0)$ state to gain more detailed understanding of the transitions. Here n is odd if electrons are fermions, and n is even if electrons are bosons. The electrons form a $\nu = 1/n$ Laughlin state in each layer. The quasiparticle labeled by $\boldsymbol{l}^T = (-1, 1)$ is neutral and has statistics $\theta_l = 2\pi/n$. Such a neutral quasiparticle corresponds to a bound state of the charge $-e/n$ Laughlin quasiparticle in one layer and the charge e/n Laughlin quasihole in the other layer. If the Laughlin quasiparticle and quasihole in the two layers have strong enough attraction (this happens when there is a strong interlayer repulsion), the quasiparticle-quasihole pair [i.e., the neutral quasiparticle labeled by $\boldsymbol{l}^T = (-1, 1)$ and the neutral antiquasiparticle labeled by $\boldsymbol{l}^T = (1, -1)$] will be spontaneously generated. This will cause a phase transition.

To understand the nature of the transition, let us first assume that the neutral (anti)quasiparticles are bosons with $\theta = 0$. The dynamical properties of the neutral (anti)quasiparticles can be modeled by a lattice-boson system which only allows 0, 1, or 2 bosons per site. The average boson density is one boson per site. In the lattice-boson model, an empty site corresponds to the neutral antiquasiparticle, a doubly occupied site corresponds to the neutral quasiparticle, and a singly occupied site correspond to no quasiparticle. Let us first ignore the boson hopping. When the interlayer repulsion is much weaker than the intralayer repulsion, the ground state of the lattice-boson model has exactly one boson per site, and is a Mott insulator. If the interlayer repulsion is much stronger than the intralayer repulsion, the lattice-boson model has two degenerate ground states, one has two bosons per site and the other has no boson. This corresponds to a charge unbalanced state, where electrons have different densities in the two layers. Such a charge unbalanced state has been observed in experiments [11]. In the presence of boson hopping, before going into the charge unbalanced phase, the boson Mott insulator must undergo a continuous phase transition into a boson superfluid phase. [The effective theory near the boson Mott insulator to superfluid transition is given by Eq. (5) without the gauge fields.] This is the phase transition between the K and K' states. In the boson condensed phase, the bosons condense into a state described by a constant wave function $\Phi(\{z_i\};\{w_i\}) = \text{const}$, where the complex number z_i (w_i) are coordinates of the neutral quasiparticles (antiquasiparticles).

When $\theta \neq 0$, the phase diagram is similar to the one for $\theta = 0$ (at least in the large N limit [4]), and the picture discussed above still applies. The effective theory near the transition is given by Eq. (5) with the CS term. In general the (anti)quasiparticle can condense into a state described by wave function

$$\prod_{i<j}[(z_i - z_j)(w_i - w_j)]^{n_e-\theta/\pi}\prod_{i,j}(z_i^* - w_j^*)^{n_e-\theta/\pi}, \tag{7}$$

where n_e is a certain even integer which corresponds to the n_e in Eq. (5).

In the dilute limit, the state with minimal $|n_e - \frac{\theta}{\pi}|$ is expected to have the lowest energy. Therefore, if $n \geq 2$, as we increase the interlayer repulsion, the $(nn0)$ state will transform into a QH state with

$$K_3 = \begin{pmatrix} n & 0 & 1 \\ 0 & n & -1 \\ 1 & -1 & 0 \end{pmatrix}, \qquad \boldsymbol{q}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{8}$$

via a continuous transition.

What is the $(K_3, \boldsymbol{q}_3)$ state? It is nothing but the $(K, \boldsymbol{q}) = (2n, 2)$ state, i.e., the $\nu = 1/2n$ Laughlin state of charge-$2e$ bosons (the Hall conductance is $\sigma_{xy} = \nu(2e)^2/h$). This is because, after a $SL(3, Z)$ transformation, the $(K, \boldsymbol{q})$ in Eq. (8) can be rewritten as

$$K = \begin{pmatrix} 2n & 0 & 0 \\ 0 & n\%2 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \boldsymbol{q} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \tag{9}$$

where $n\%2 = 1$ if $n = \text{odd}$, and $n\%2 = 0$ if $n = \text{even}$. According to Haldane's topological instability [9,10], Eq. (9) just describes the $(K, \boldsymbol{q}) = (2n, 2)$ state. On the edge, the three edge branches described by K_3 in Eq. (9) can be viewed as the reconstructed edge of the $(K, \boldsymbol{q}) = (2n, 2)$ state [10].

When $n = 2$, the effective theory [Eq. (5)] can be mapped into a free fermion model (since the interaction terms are irrelevant)

$$\mathcal{L} = \psi^\dagger \partial_\mu \tau^\mu \psi - m\psi^\dagger \tau^3 \psi. \tag{10}$$

The effective free fermion theory allows us to calculate all of the physical properties of the transition, and we can show rigorously that the transition is continuous.

The effective theory [Eq. (5)] for the transition has a U(1) symmetry—the conservation of the neutral quasiparticle ϕ. However, such a U(1) symmetry is broken by interlayer electron tunneling which creates n neutral (anti)quasiparticles. The electron tunneling operator has a form $\hat{M}\phi^n$, where $\hat{M}$ is an operator that creates one unit of a_μ flux. Note that the combination $\hat{M}\phi^n$ is gauge invariant and the effective Hamiltonian/Lagrangian equation (5) should contain a term $t\hat{M}\phi^n + \text{H.c.}$ to describe the electron interlayer tunneling.

When n is large, we expect the tunneling term $t\hat{M}\phi^n + \text{H.c.}$ to be irrelevant, and it can affect the properties of the transition only if t is large. When n is small the effect of the tunneling term may be important even in the small t limit. When the tunneling term is important, its effect is hard to study. However, since the effective theory for $n = 2$ can be mapped into a free fermion model, the effect of the tunneling term on the transition can be studied exactly. This situation has been studied in Refs. [5,6] for a related (331) state [or a more general $(q + 1, q + 1, q - 1)$ state in Ref. [5]]. In the following we will show how to make contact with their derivation.

For $n = 2$, we start with the fermionic effective theory [Eq. (10)]. Since the electron tunneling operator creates a pair of quasiparticles ψ, it has a form $\psi^T\tau^2\psi$. Thus the effective Lagrangian with tunneling can be written as

$$\mathcal{L} = \psi^\dagger \partial_\mu \tau^\mu \psi - m\psi^\dagger \tau^3 \psi + (t\psi^T\tau^2\psi + \text{H.c.}), \tag{11}$$

where t is the amplitude of the interlayer electron tunneling. After diagonalization, the Hamiltonian becomes $H = \sum_{\boldsymbol{k},\alpha=\pm} E_\alpha(\boldsymbol{k})\lambda^\dagger_{\alpha,\boldsymbol{k}}\lambda_{\alpha,\boldsymbol{k}}$ with

$$E_\pm(\boldsymbol{k}) = \sqrt{k^2 + m^2 + |t|^2 \pm 2\sqrt{m^2|t|^2 + k^2(\text{Im}t)^2}}. \tag{12}$$

The system contains gapless excitations (i.e., reaches a critical point) when $m = |t|$ or $m = -|t|$. We see that the single transition point is split into two transition points by the tunneling term. The new phase diagram is sketched in Fig. 2. Near the new transition points the low energy excitations are described by one free gapless Majorana fermion $H = \sum_{\boldsymbol{k}} \sqrt{v^2\boldsymbol{k}^2 + (|t| - |m|)^2}\lambda^\dagger_{\boldsymbol{k}}\lambda_{\boldsymbol{k}}$, where $v = 1 - \frac{(\text{Im}t)^2}{|mt|}$. This agrees with the result obtained in Refs. [5,6]. Again, all of the physical properties of the transition can be calculated from the above free fermion effective theory. The state between the two new transition points [5,6] is a non-Abelian Pfaffian state (for bosons) proposed by Moore and Read [12]. It was suggested [13] that such a p-wave paired state may describe the $\nu = 5/2$ state observed in experiments.

From the phase diagram Fig. 2 (which has been given in Ref. [5]), we see that the continuous transition from the (220) state to the non-Abelian Pfaffian state can also be induced by increasing the interlayer tunneling t. One

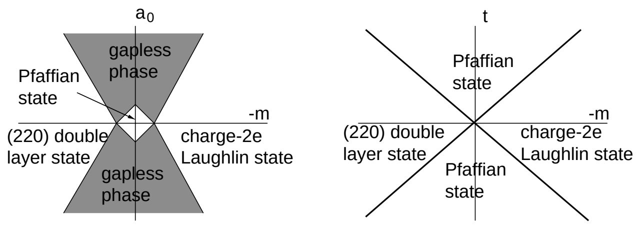

FIG. 2. The phase diagram for the (220) state in the presence of interlayer tunneling. We have assumed $\mathrm{Im}t = 0$. The left diagram has a fixed t and the right one has $a_0 = 0$.

naturally asks what type of QH state can a large t induce from the $(nn0)$ state?

We first note that the analytic part of the $(nn0)$ ground state wave function can be written as a correlation function of $(2 + 0)D$ chiral fermions [12,14]. The chiral fermions ψ_i are defined by operator product expansion (OPE) $\psi_i^\dagger(z)\psi_j(0) = \delta_{ij}/z$, where $z = x + iy$. By introducing electron operators in the two layers $\psi_{e1} = \prod_{i=1}^{n} \psi_i$ and $\psi_{e2} = \prod_{i=n+1}^{2n} \psi_i$, the $(nn0)$ wave function can be written as $\Phi_{(nn0)}(\{z_i\};\{w_i\}) = \langle e^{-iN\phi} \prod_{i=1}^{N} \psi_{e1}(z_i)\psi_{e2}(w_i)\rangle$, where the boson field ϕ is defined by $\frac{1}{2\pi}\partial_z\phi(z) = \sum_{i=1}^{2n} \psi_i^\dagger(z)\psi_i(z)$.

For the (220) state, after the transition, the wave function of the resulting Pfaffian state is just the (220) wave function symmetrized between z_i and w_j. Such a wave function can be written as a correlation function of a single electron operator $\psi_e = \psi_{e1} + \psi_{e2}$:

$$\Phi_{\mathrm{nab}}(z_1, \ldots, z_{2N}) = \left\langle e^{-iN\phi} \prod_{i=1}^{2N} \psi_e(z_i) \right\rangle. \qquad (13)$$

Since only a single electron operator is involved, Φ_{nab} is a single layer state. This is consistent with the fact that the Φ_{nab} state is induced by a large interlayer tunneling.

Here we would like to conjecture that a similar phenomenon will also happen for the $(nn0)$ state with $n > 2$. A large interlayer tunneling t will change the $(nn0)$ state into the Φ_{nab} state defined in Eq. (13).

When $n = 3$, we find that the Φ_{nab} state is a non-Abelian state. The edge excitations are generated by ψ_e's and $\psi_e^\dagger$'s [14]. Through the OPE of ψ_e and $\psi_e^\dagger$, one can show that the neutral edge excitations are generated by ρ_+, ρ_-^2 and $\cos(3n\phi_-)$, where $\rho_\pm$ are the currents in $\mathrm{U}(1)^2$ Kac-Moody algebra and $\phi_\pm$ are defined by $\partial_z\phi_\pm/2\pi = \rho_\pm$. Physically, ρ_+ corresponds to the total electron density and ρ_- corresponds to the difference of the electron densities in the two layers. They have the following OPE: $\rho_\alpha(z)\rho_\beta(0) = 2\delta_{\alpha\beta}/3z^2$. The electron operator can be written as $\psi_e = e^{i3\phi_+/2}\cos(3\phi_-/2)$ through bosonization. Thus the electron propagator along the edge has an exponent 3: $\langle\psi_e\psi_e^\dagger\rangle \sim z^{-3}$. The quasiparticle operators must be mutually local with respect to the electron operator [3]. We find that the quasiparticles with the lowest charge are created by $\psi_q = e^{i\phi_+/2}\cos(\phi_-/2)$ which carries charge $e/3$. The quasiparticle propagator has an exponent $1/3$. The electron and the quasiparticle exponents and charges (and hence the edge tunneling I-V curve) for our single layer $\nu = 2/3$ non-Abelian state are identical with the $\nu = 1/3$ Laughlin state.

Since all of the charged excitations remain to have finite energy gap across the transition, the continuous topological phase transitions discussed in this paper may not be easy to observe. Notice that the neutral gapless excitations at the transition carry an electric dipole moment in the z direction. Thus one way to detect them is to use surface acoustic phonons. Also, the edge states before and after the transition are very different. Near the transition, the velocity of one edge mode approaches zero, and such a mode becomes the gapless bulk excitation at the transition. Thus the transition should also be detectable through edge tunneling experiments.

We would like to mention that the phase diagram for the (n, n, m) state is the same as that of the $(n - m, n - m, 0)$ state discussed above. The neutral excitations in the two states are identical. As a consequence, the critical theories for the transitions in the two states are identical. The results in this paper can be easily generalized to any (mnl) bilayer states and hierarchical states.

X. G. W. is supported by NSF Grant No. DMR-97-14198 and by NSF-MRSEC Grant No. DMR-98-08941.

[1] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[2] R. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[3] X.-G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990); X.-G. Wen, Adv. Phys. **44**, 405 (1995); X.-G. Wen and Y. S. Wu, Nucl. Phys. **B419**, 455 (1994).
[4] X.-G. Wen and Y.-S. Wu, Phys. Rev. Lett. **70**, 1501 (1993).
[5] N. Read and D. Green, cond-mat/9906453.
[6] M. P. A. Fisher, T. Senthil, and X.-G. Wen (to be published).
[7] B. Blok and X.-G. Wen, Phys. Rev. B **42**, 8133 (1990); **42**, 8145 (1990); N. Read, Phys. Rev. Lett. **65**, 1502 (1990); J. Fröhlich and T. Kerler, Nucl. Phys. **B354**, 369 (1991); J. Fröhlich and A. Zee, Nucl. Phys. **B364**, 517 (1991); X.-G. Wen and A. Zee, Phys. Rev. B **46**, 2290 (1992).
[8] W. Chen, M. P. A. Fisher, and Y.-S. Wu, Phys. Rev. B **48**, 13 749 (1993).
[9] F. D. M. Haldane, Phys. Rev. Lett. **74**, 2090 (1995).
[10] H.-C. Kao, C.-H. Chang, and X.-G. Wen, cond-mat/9903275.
[11] H. C. Manoharan *et al.*, Phys. Rev. Lett. **79**, 2722 (1997).
[12] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[13] M. Greiter, X.-G. Wen, and F. Wilczek, Phys. Rev. Lett. **66**, 3205 (1991); Nucl. Phys. **B374**, 567 (1992).
[14] X.-G. Wen, Y.-S. Wu, and Y. Hatsugai, Nucl. Phys. **B422**, 476 (1994).