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Linking Numbers for Self-Avoiding Loops and Percolation: Application to the Spin Quantum Hall Transition

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Nonlocal twist operators are introduced for the O(n) and Q-state Potts models in two dimensions which count the numbers of self-avoiding loops (respectively, percolation clusters) surrounding a given point. Their scaling dimensions are computed exactly. This yields many results: for example, the number of percolation clusters which must be crossed to connect a given point to an infinitely distant boundary. Its mean behaves as $(1/3\sqrt{3}\,\pi)\,|\ln(p_c - p)|$ as $p \to p_c-$. As an application we compute the exact value $\sqrt{3}/2$ for the conductivity at the spin Hall transition, as well as the shape dependence of the mean conductance in an arbitrary simply connected geometry with two extended edge contacts.

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The conformal field theory/Coulomb gas approach to two-dimensional percolation and self-avoiding walk problems has been extraordinarily fruitful [1–3]. In addition to values for many of the critical exponents, other universal scaling functions such as percolation crossing probabilities have been obtained exactly [4,5]. In this Letter a set of correlation functions of nonlocal operators is introduced, which describe topological properties of percolation clusters and self-avoiding walks, and count the number of clusters or loops which must be crossed in order to connect two or more points.

It turns out that these exponents may easily be computed using standard Coulomb gas methods, for general n or Q in the O(n) or Q-state Potts model, respectively. In the limits $n \to 0$ or $Q \to 1$, corresponding to self-avoiding walks or to percolation, respectively, the scaling dimensions of these operators vanish, so that some of their correlations are trivial. However, it is their *derivatives* with respect to n or Q which give physical information, thereby giving rise to a variety of logarithmic behavior. The frequent occurrence of logarithmic correlations in such conformal field theories (CFTs) with vanishing central charge has recently been pointed out in several contexts [6,7].

In certain cases these topological operators may be recognized as *twist* operators which have already been identified in $c = 1$ theories [8] and the three-state Potts model [9]. From the CFT point of view, they turn out to correspond to degenerate Virasoro representations labeled by $(r,s) = (1,2)$ in the Kac classification. These bulk operators have not previously been identified for general n and Q. The fact that they are degenerate means that their higher-point correlations may be computed exactly.

In addition to bulk operators of the above type it is also possible to define operators which count the number of loops or clusters surrounding a point near the boundary of a system. They may be used to compute the universal mean conductance at the spin quantum Hall transition, which has recently been shown to map onto the percolation problem [10].

O(n) *model.*—Let us recall the elements of the Coulomb gas approach [1]. n-component spins $\mathbf{s}(r)$ with $\mathbf{s}^2(r) = 1$ are placed at the sites r of a lattice, with a nearest neighbor interaction. The partition function $\mathrm{Tr}\prod_{rr'}[1 + y\mathbf{s}(r)\cdot\mathbf{s}(r')]$, when expanded in powers of y, gives a sum over self-avoiding loops with a factor y for each bond and n for each loop. Each loop may be replaced by a sum over its orientations, with a weight $e^{\pm i\pi\chi}$ for each, with χ chosen so that the sum gives $n = 2\cos\pi\chi$. This gas of oriented loops is then mapped onto a height model with variables $\phi(R) \in \pi\mathbf{Z}$ on the dual lattice, such that on the dual bond RR' $\phi(R) - \phi(R') = 0, \pm 1$ according to whether the bond it crosses is empty or is

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occupied by a loop segment of one orientation or the other. At the critical fugacity y_c this is supposed to renormalize onto a free field with reduced free energy functional $[g/(4\pi)]\int(\partial\phi)^2 d^2r$, with respect to which the long-distance behavior of all correlations may be computed as long as phase factors associated with noncontractible loops are correctly accounted for. g may be fixed by a variety of methods: for the dilute regime of interest here, $g = 1 + \chi$. On an infinitely long cylinder of perimeter L, in order to correctly count loops which wrap around the cylinder with weight n it is necessary to insert factors $e^{\pm i\chi/\phi(\pm\infty)}$ at either end: these modify the free energy per unit length to $-(\pi/6L)[1 - (6/g)\chi^2]$, from which the value of the central charge $c = 1 - 6(g-1)^2/g$ follows. For self-avoiding walks, $n = 0$, $\chi = \frac{1}{2}$, and $g = \frac{3}{2}$, so that $c = 0$.

Now suppose that loops which wrap around the cylinder are counted with a different weight $n' = 2\cos\pi\chi'$. (A similar construction was used in Ref. [11] to count the winding angle of open walks.) The cylinder free energy per unit length will be modified by a term $(\pi/g)(\chi'^2 - \chi^2)$. In the plane, this may be interpreted as the insertion of a nonlocal operator $\sigma_{n'}$ whose scaling dimension is

$$x(n;n') = [1/2g(n)](\chi'^2 - \chi^2), \qquad (1)$$

so that the two-point function $\langle\sigma_{n'}(r_1)\sigma_{n'}(r_2)\rangle$ decays like $|r_1 - r_2|^{-2x(n;n')}$ at criticality. The interpretation of this is as a *defect line* joining r_1 and r_2: loops which cross this defect line an odd number of times, that is, which surround either but not both of the points [12], now carry a factor n' instead of n. A particular example is $n' = -n$, which is equivalent to making the identification $\mathbf{s}(r) \to -\mathbf{s}(r)$ as the defect line is crossed. Such *twist* operators have been identified in other models [8,9]: like disorder operators, they reflect the existence of nontrivial homotopy. From (1) it follows that the dimension of this operator is $x(n;-n) = (3/2g) - 1$, which is $x_{1,2}$ in the Kac classification $x_{r,s} = [(rg - s)^2 - (g-1)^2]/2g$. However, for small n and n', $x(n,n') \sim (1/6\pi)(n - n')$, so that, to first order, it does not matter whether $x(n,-n)$ or $x(0;n')$ is used apart from factors of 2. Note that if $n' = -n$ the fusion rules of CFT imply the operator product expansion (OPE) $\sigma\cdot\sigma = 1 + \epsilon + \ldots$, where ϵ is the $(1,3)$ operator which has been identified as the local energy density of the O(n) model [2]. $\epsilon(r)$ counts whether a given bond r is occupied or not—consistent with the interpretation that, as $a \to 0$, the O(n) term in $\sigma(r+a)\sigma(r-a)$ counts whether the single loop is trapped between the points $r \pm a$.

Here are a few examples of the application of this formula. Away from criticality it implies that $\langle\sigma_{n'}(R)\rangle \sim C(n;n')\xi^{-x(n,n')}$, where the correlation length ξ behaves as $(y_c - y)^{-\nu}$, and $C(0;0) = 1$. When $n = 0$, the coefficient of n'^M counts configurations of M nested loops surrounding the point R of the dual lattice. Denoting the number of such loops of *total* length N by $b_N^{(M)}$, the most singular term in the generating function $\sum_N b_N^{(M)} y^N \sim (1/M!)[(1/8\pi)|\ln(y_c - y)|]^M$ as $y \to y_c-$, using $\nu = \frac{3}{4}$. This yields the asymptotic behavior

$$b_N^{(M)} \sim [\sigma_0/(M-1)!](1/8\pi)^M (1/N)(\ln N)^{M-1}\mu^N,$$

where $\mu = y_c^{-1}$ is the usual lattice-dependent connective constant, and σ_0 is a positive integer taking account of the fact that, on a loose-packed lattice, there are σ_0 equivalent singularities on the circle $|y| = y_c$. For $M = 1$, one may sum over the position of the marked point R, instead of summing over the position of the loop, so that $b_N^{(1)} = A_N p_N$, where p_N is the number of single loops per lattice site and A_N is their average area. The amplitude $1/8\pi$ in this case agrees with an earlier result found by a different (more involved) method [13]. It is interesting also to calculate higher point functions. For example the O(n) term in the *connected* four-point function $\langle\sigma\sigma\sigma\sigma\rangle_c$ counts the number of single loops which have nontrivial winding around all four points. Examples are shown in Fig. 1. Since σ is a (1,2) operator this four-point function may be computed exactly at criticality, in terms of hypergeometric functions. The details will be given elsewhere [14].

Percolation.—The Coulomb gas formulation of the Q-state Potts model is similar to the above, except that it is valid only at the critical point [1]. The model is first mapped onto the random cluster model, in which every cluster configuration is weighted by a factor of $p/(1-p)$ for each bond and Q for each cluster. Each cluster may be identified by its outer and inner hulls, which form a dense set of closed loops. At criticality, each hull then carries a weight $\sqrt{Q}$. Once again this may be mapped onto a gas of oriented loops with phase factors $e^{\pm\pi\chi}$, where $\sqrt{Q} = 2\cos\pi\chi$, and then to a free field theory, where, however, this time $g = 1 - \chi$. The central charge then vanishes for $g = \frac{2}{3}$, corresponding to $\chi = \frac{1}{3}$ or $Q = 1$, the percolation limit. Once again one may define a nonlocal operator which counts the hulls which surround a marked point with a different weight $\sqrt{Q'} = 2\cos\pi\chi'$, and standard Coulomb gas methods then lead to a result identical in form to (1) for its dimension $x(Q;Q')$. For this to be given by $x_{(1,2)} = (3g/2) - 1$, $Q' = (2-Q)^2$.

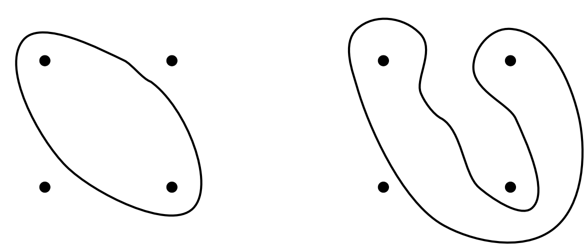

FIG. 1. Examples of loops which wind nontrivially around four points, counted by the connected four-point function $\langle\sigma\sigma\sigma\sigma\rangle_c$.

For $Q = 3$, for example, the clusters which wrap around the marked point are counted with weight $Q' = 1$. This is consistent [15] with the identification of the $\mathbf{Z}_2$ twist operator for the three-state model made in Ref. [9]. For the Ising model ($Q = 2$), it is the same as the disorder operator. Note that as $Q \to 1$, the $O(Q - 1)$ term in $\sigma(r + a)\sigma(r - a)$ counts the *number* of clusters separating $r \pm a$, as compared with the $O(n)$ model where it simply counted whether or not the single loop passed between them. For this reason the lattice interpretation of the (1,3) operator in the OPE $\sigma \cdot \sigma$ is problematic in this case [16].

For $Q = 1 + \delta$ and $Q' = 1 + \delta'$ ($|\delta|, |\delta'| \gg 1$) note that $x(Q;Q') \sim (1/4\sqrt{3}\,\pi)(\delta - \delta')$ (with $\delta' \sim -2\delta$ corresponding to $x_{1,2}$), while for small Q' $x(1;Q') = \frac{5}{48} - \frac{3}{8\pi}\sqrt{Q'} + O(Q')$. As before, one application of these results is to count the number of clusters which surround a given point, that is, which have to be crossed to escape to infinity. If $P(M)$ is the probability of there being exactly M such clusters,

$$\sum_M P(M) Q'^M \sim C(Q')(p_c - p)^{\nu x(1;Q')},$$

where now $\nu = \frac{4}{3}$. For $Q' = 0$ this gives $P(0)$, which is the probability that the dual site R is connected to the boundary by a set of dual bonds: this gives the usual exponent $\beta = \frac{5}{36}$ of percolation. Expanding in powers of $\sqrt{Q'}$ now gives

$$P(M) \sim P(0)(|\ln(p_c - p)|/2\pi)^{2M}/(2M)!$$

Note that although the amplitude in $P(0)$ is not universal, the ratios $P(M)/P(0)$ are.

However, this is valid only for $M \ll |\ln(p_c - p)|$. To find the behavior near the average value, expand around $Q' = 1$. The mean value $\overline{M} \sim (1/3\sqrt{3}\,\pi)|\ln(p_c - p)|$, and the variance $\overline{M^2} - \overline{M}^2 \propto |\ln(p_c - p)|$ also. This implies that, as $p \to p_c-$, the distribution of M becomes peaked about $\overline{M}$. Alternatively, one could work at the percolation threshold in a finite system of size L, in which case $\overline{M} \sim (1/4\sqrt{3}\,\pi)\ln L$.

Spin quantum Hall transition.—Recently Gruzberg *et al.* [10] have shown that certain properties of a model of noninteracting quasiparticles for the spin Hall transition (a 2D metal-insulator transition in a disordered system in which time-reversal symmetry is broken but SU(2) spin-rotation symmetry is not [17]) may be mapped exactly onto percolation. In particular, the mean conductance between two extended contacts on the boundary of a finite system is (apart from a factor of 2 for the spin sum) equal to the mean number of distinct clusters whose outer hulls connect the two contacts. As argued above, such quantities are related to the derivative with respect to Q at $Q = 1$ of correlation functions of a twist operator. This will however now be a *boundary* twist operator. While it is possible to adapt the above Coulomb gas arguments to account for the boundary, since in other examples such methods are known to fail for boundary operators, we instead give a more direct argument.

First consider the example of an annulus of width L and circumference W. The geometry is shown in Fig. 2. Apart from the clusters whose outer hulls cross the sample, there are those which touch the lower edge but not the upper, those which do the opposite, and those which touch neither edge. There may also be one cluster which crosses the sample but which also wraps around the annulus, so that its outer hulls do not connect the contacts. Denote the numbers of such clusters in a given configuration of the random cluster version of the Potts model by N_c, N_1, N_2, N_b, and N_w, respectively. (Note that $N_c = 0$ if $N_w = 1$). Let $Z_{ij}(Q)$ denote the Potts model partition function with boundary condition of type i on the lower edge and j on the upper edge. The cases of interest are where i or j correspond to either free boundary conditions, denoted by f, or to fixed, in which the Potts spins on the boundary are frozen into a given state, say 1. Then

$$Z_{ff} = \langle Q^{N_c + N_w + N_1 + N_2 + N_b} \rangle \qquad Z_{11} = \langle Q^{N_b} \rangle,$$
$$Z_{1f} = \langle Q^{N_2 + N_b} \rangle \qquad Z_{f1} = \langle Q^{N_1 + N_b} \rangle,$$

so that

$$\langle N_c + N_w \rangle = (\partial/\partial Q)|_{Q=1}(Z_{ff}Z_{11}/Z_{f1}Z_{1f}). \quad (2)$$

According to the theory of boundary CFT [18], $Z_{ij} \sim \exp\{\pi[(c/24) - \Delta_{ij}](W/L)\}$ as $W/L \to \infty$, where Δ_{ij} is the lowest scaling dimension out of all the conformal blocks which can propagate around the annulus with the given boundary conditions. When $i = j$ this corresponds to the identity operator, so that $\Delta_{ii} = 0$, but for the mixed case $(ij) = (f1)$ it corresponds to the (1,2) Kac operator, so that $\Delta_{f1} = \Delta_{1,2} = \frac{1}{2}x_{1,2}(Q)$ in the previous notation. This identification was previously used at $Q = 1$ in Ref. [4] to compute crossing probabilities, i.e., the probability that $N_c > 0$, in simply connected regions. Substituting into (2) gives $\langle N_c \rangle \sim 2\pi\Delta'_{1,2}(1)(W/L)$ as $W/L \to \infty$, since $N_w \leq 1$. From this follows the universal critical conductivity $\sqrt{3}/2$.

At finite W/L the corrections to the mean conductance are expected to be of the order of $e^{-\pi\Delta_{2,2}W/L}$, where $\Delta_{2,2} = \frac{1}{8}$ at $Q = 1$, but the full dependence requires knowledge of the entire operator content of the model for the different

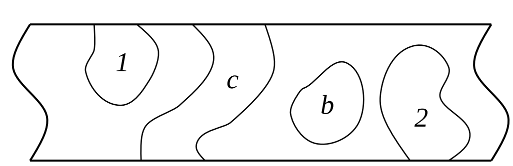

FIG. 2. Annular geometry with contacts along either edge. Periodic boundary conditions are implied in the horizontal direction. Examples are shown of Potts clusters of types c, 1, 2, and b. The mean conductance is proportional to the mean number of type c.

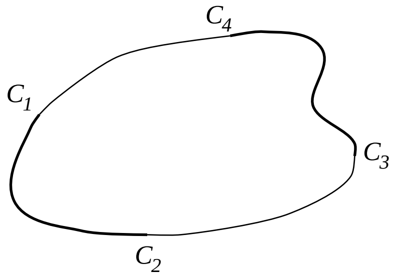

FIG. 3. Simply connected region with contacts C_1C_2 and C_3C_4 along its edge.

boundary conditions. This seems to be beyond the reach of current methods. However, for a *simply connected* finite sample the arguments of Ref. [4] may be adapted. Consider a simply connected region with contacts C_1C_2 and C_3C_4 on its boundary, as shown in Fig. 3. The remainder of the boundary has hard wall conditions on the quasiparticle wave functions, corresponding to free boundary conditions on the Potts spins. The mean number of clusters crossing between the contacts is still given by (2) (with $N_w = 0$), where the different boundary conditions are placed on the segments C_1C_2 or C_3C_4, with the remaining boundary Potts spins being free. This may then be written in terms of correlation functions of boundary condition changing operators [18]

$$\langle N_c \rangle = \frac{\partial}{\partial Q}\Big|_{Q=1} \left(\frac{\langle \phi_{f1}(C_1)\phi_{1f}(C_2)\phi_{f1}(C_3)\phi_{1f}(C_4)\rangle}{\langle \phi_{f1}(C_1)\phi_{1f}(C_2)\rangle\langle \phi_{f1}(C_3)\phi_{1f}(C_4)\rangle} \right).$$

These correlation functions are computed by conformally mapping the interior of the region to the upper half plane. Any conformal rescaling factors for $Q \neq 1$ cancel in the ratio, which then depends only on the cross ratio $\eta = (z_1 - z_2)(z_3 - z_4)/(z_1 - z_3)(z_2 - z_4)$ of the images z_i of the points C_i under this mapping. For a rectangle with $|C_1C_2| = W$ and $|C_2C_3| = L$, $\eta = (1 - k)^2/(1 + k)^2$, where $W/L = K(1 - k^2)/2K(k^2)$ and K is the complete elliptic integral of the first kind. Since ϕ_{f1} is a degenerate (1,2) operator, its four-point function satisfies a hypergeometric equation. The details of this calculation will be given elsewhere [14]. The result for the mean conductance is

$$\bar{g} = 1 - \frac{\sqrt{3}}{2\pi} \left(\ln(1 - \eta) + 2 \sum_{m=1}^{\infty} \frac{(\frac{1}{3})_m}{(\frac{2}{3})_m} \frac{(1 - \eta)^m}{m} \right).$$

For $W/L \gg 1$ this reproduces the above result for the conductivity. In the opposite limit $\bar{g} \sim Ae^{-(\pi/3)(L/W)}$, in agreement with Ref. [10], but now with a definite prefactor $A = 3\Gamma(\frac{2}{3})/2\Gamma(\frac{1}{3})^2$.

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