Universal Teleportation with a Twist

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We give a transfer theorem for teleportation based on *twisting* the entanglement measurement. This allows one to say what local unitary operation must be performed to complete the teleportation in any situation, generalizing the scheme to include overcomplete measurements, non-Abelian groups of local unitary operations (e.g., angular momentum teleportation), and the effect of nonmaximally entangled resources.

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One of the most profound results of quantum information theory is the discovery of quantum teleportation protocols [1–4]. Teleportation is the disembodied transport of quantum states between subsystems through a classical communication channel requiring a shared resource of entanglement. The demonstration of teleportation elevates entanglement from a perennial theoretical chestnut to a practical resource. The details of protocols for teleportation may vary; specification of subsystems, the shared entangled state, and the description of joint measurements at the sender (Alice) or receiver (Bob). For example, already there have been several experimental implementations of teleportation [5–7] and other protocols have been proposed [8]. We show in this paper that all teleportation schemes can be cast in a common form with generalized (overcomplete) measurements and which enables us to identify the local unitary operations required to complete a teleportation scheme.

Let us start by recalling a maximally entangled state in $\mathcal{H} \otimes \mathcal{H}$

$$|\Psi\rangle\!\rangle = \frac{1}{\sqrt{d}} \sum_n e^{-i\phi_n} |n\rangle \otimes |n\rangle , \tag{1}$$

where $\{|n\rangle\}$ is any basis in $\mathcal{H}$ and d is the dimension of $\mathcal{H}$ (the infinite dimensional case will be considered later). In the following we will adopt the notation: use double ket (bra) $|\cdots\rangle\!\rangle$ to denote vectors in $\mathcal{H} \otimes \mathcal{H}$ and customary single ket (bra) for vectors in $\mathcal{H}$.

It is now well known that we may span the set of *all* maximally entangled states by local unitary operations. Therefore, it is sufficient to consider local unitary operators acting only on one space. In what follows we shall use a *twist* operation which swaps a pair of particles. Here we introduce a "democratic" notation so that given a pair of systems in a state

$$|\overrightarrow{\Psi}\rangle\!\rangle = \sum_{jk} c_{jk} |j\rangle \otimes |k\rangle , \tag{2}$$

the twisted/swapped version is denoted by

$$|\overleftarrow{\Psi}\rangle\!\rangle = \sum_{jk} c_{jk} |k\rangle \otimes |j\rangle , \tag{3}$$

and *vice versa* $|\overleftarrow{\Psi}\rangle\!\rangle \leftrightarrow |\overrightarrow{\Psi}\rangle\!\rangle$. The generalization to mixed states follows trivially. A final piece of notation we introduce is the *transfer* operator [9]

$$\mathcal{T}_{ba} = \sum_n |n\rangle_{ba}\langle n| . \tag{4}$$

We now give several identities which succinctly describe teleportation. For an arbitrary *maximally* entangled state $|\overleftarrow{\Psi}\rangle\!\rangle$ one can easily show that

$${}_{1,2}\langle\!\langle \overrightarrow{\Psi} \,|\, \overleftarrow{\Psi} \rangle\!\rangle_{2,3} = \frac{1}{d}\, \mathcal{T}_{31} , \tag{5}$$

where the subscripts denote the particle numbers of each of the states and operators involved.

Equation (5) implies

$${}_{1,2}\langle\!\langle \overrightarrow{\Psi} | [|\phi\rangle_1 \otimes |\overleftarrow{\Psi}\rangle\!\rangle_{2,3}] = \frac{1}{d}\, |\phi\rangle_3 , \tag{6}$$

where $|\phi\rangle$ is an arbitrary (unknown) quantum state; also for linearity

$${}_{2,3}\langle\!\langle \overrightarrow{\Psi} | [|\Phi\rangle\!\rangle_{1,2} \otimes |\overleftarrow{\Psi}\rangle\!\rangle_{3,4}] = \frac{1}{d}\, |\Phi\rangle\!\rangle_{1,4} , \tag{7}$$

and similarly

$${}_{2,3}\langle\!\langle \overrightarrow{\Psi} | [|\overleftarrow{\Psi}\rangle\!\rangle_{1,2} \otimes |\Phi\rangle\!\rangle_{3,4}] = \frac{1}{d}\, |\Phi\rangle\!\rangle_{1,4} , \tag{8}$$

which will correspond to *entanglement swapping* for an arbitrary unknown (entangled) two-mode state $|\Phi\rangle\!\rangle$. Some other trivial variations of these identities are

$${}_{1,2}\langle\!\langle \overleftarrow{\Psi} \,|\, \overrightarrow{\Psi} \rangle\!\rangle_{2,3} = \frac{1}{d}\, \mathcal{T}_{31} , \tag{9}$$

and

$${}_{2,3}\langle\!\langle \overrightarrow{\Psi} \,|\, \overleftarrow{\Psi} \rangle\!\rangle_{1,2} = {}_{2,3}\langle\!\langle \overleftarrow{\Psi} | |\overrightarrow{\Psi} \rangle\!\rangle_{1,2} = \frac{1}{d}\, \mathcal{T}_{13} , \tag{10}$$

and other identities analogous to Eqs. (6) and (7) follow.

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Let us see how these identities allow us to understand teleportation. Start with an unknown state and a shared arbitrary maximally entangled resource $|\phi\rangle_1 \otimes |\Phi\rangle\rangle_{2,3}$. Perform a measurement on the first two subsystems yielding a maximally entangled result $|\overrightarrow{\Psi}\rangle\rangle_{1,2}$. We emphasize that this measurement may be complete or overcomplete. Information about which entangled state was found by Alice is transmitted to Bob. To complete the teleportation protocol Bob must convert $|\Phi\rangle\rangle_{2,3}$ into the twisted version of the entangled state actually found by Alice, i.e., $|\overleftarrow{\Psi}\rangle\rangle_{2,3}$. This conversion involves a local unitary operation which now leaves us with the situation described by Eq. (6). Using it shows that the initial unknown state at Alice's end has been successfully transferred to Bob.

At this point it is worthwhile stepping back and looking at what this teaches us about quantum teleportation. In the ideal case Alice and Bob must share a maximally entangled state and Alice must be able to perform a measurement which yields a maximally entangled state. The details of the measurement, for example, whether it involves projection measurements or a positive operator valued measure (POVM) is unimportant. The reconstruction operation relies only on Bob being able to locally convert his shared entanglement into the swapped version that Alice found. But this is a general feature of maximally entangled states. In fact, it lies at the heart of several other quantum communication protocols. In quantum dense coding this ability allows us to encode the square as many orthogonal states as are supported by the Hilbert space we are acting on [10]. This yields potentially a doubled channel capacity. Similarly, in any scheme which tries to implement bit commitment, the freedom to locally convert any maximally entangled state to any other allows Alice to cheat with impunity [11,12]. Now we have shown that this same freedom also drives the quantum teleportation protocol. This commonality improves our understanding of the ways in which the manipulation of shared entanglement may be used.

In order to interpret the vector $|\overrightarrow{\Psi}\rangle\rangle$ as the result of a measurement, we need an (over)complete set of maximally entangled vectors. This can be easily achieved by having the unitary operator $U \equiv U(g)$ as an element of a group $\mathbf{G} = \{g\}$ of transformations g with unitary irreducible representation (UIR) $U(g)$ on Alice's Hilbert space $\mathcal{H}$. Then, for *any* maximally entangled state $|\Psi\rangle\rangle$, one has the identity

$$\int_{\mathbf{G}} dg\, U(g) \otimes \mathbb{1}|\Psi\rangle\rangle\langle\langle\Psi|U^\dagger(g) \otimes \mathbb{1} = \frac{1}{d}\, \mathbb{1} \otimes \mathbb{1}\,, \quad (11)$$

which easily follows from the identity (Schur's lemma)

$$\int_{\mathbf{G}} dg\, U(g) A U^\dagger(g) = \mathrm{Tr}(A)\mathbb{1}\,, \quad (12)$$

which holds for any operator A on $\mathcal{H}$. The invariant measure dg is normalized as

$$\int_{\mathbf{G}} dg\, |\langle u|U(g)|v\rangle|^2 = 1\,, \quad (13)$$

which is true for any pair of normalized vectors $|u\rangle$, and $|v\rangle$ due to the irreducibility of the representation (assuming square integrable UIR for simplicity). Equation (11) means that the set of vectors

$$|\overrightarrow{\Psi}_g\rangle\rangle \equiv U(g) \otimes \mathbb{1}|\Psi\rangle\rangle, \qquad g \in \mathbf{G} \quad (14)$$

makes a (generally not orthogonal) POVM that represents a measurement on $\mathcal{H} \otimes \mathcal{H}$ with result g. The measurement correlates Alice's Hilbert space with the entangled resource. Alice gets the result g and communicates it to Bob classically, and as already mentioned Bob converts his shared entanglement $|\Phi\rangle\rangle_{2,3}$ into the twisted version of the entangled state found by Alice, i.e., $|\overleftarrow{\Psi}_g\rangle\rangle_{2,3} = \mathbb{1} \otimes U(g)|\Psi\rangle\rangle_{2,3}$. For each result g the state $|\phi\rangle$ is teleported according to the overall transformation

$$_{1,2}\langle\langle\overrightarrow{\Psi}_g|[|\phi\rangle_1 \otimes |\overleftarrow{\Psi}_g\rangle\rangle_{2,3}] = \frac{1}{d}\,|\phi\rangle_3\,. \quad (15)$$

For discrete groups the sum replaces the integral over $\mathbf{G}$. Mathematically, Eq. (15) represents a *pure instrument* [13], which describes the state reduction depending on the outcome g of the measurement, and sends a pure state into a pure state. In the general case such an instrument has the form

$$\frac{\Omega_x|\phi\rangle}{\|\Omega_x|\phi\rangle\|} = |\phi_x\rangle\,, \quad (16)$$

where x is the measurement outcome and $|\phi_x\rangle$ is the state conditioned by the result x. The case of teleportation is peculiar because the conditioned state is identical to the original one, independent of the measurement outcome, and on the other hand it is "teleported" to another space. In such a scenario the teleportation map should be regarded in the following way:

$$\frac{\Omega_g|\phi\rangle_1}{\|\Omega_g|\phi\rangle_1\|} = \mathcal{T}_{31}|\phi\rangle_1\,, \quad (17)$$

where $\Omega_g = {}_{1,2}\langle\langle\overrightarrow{\Psi}_g|\overleftarrow{\Psi}_g\rangle\rangle_{2,3} \equiv \frac{1}{d}\mathcal{T}_{31}$. Notice that Eq. (11) has the relevant feature that phase factors in the group composition law can be neglected. In mathematical terms this means that if the unitary representation is of the "projective" form

$$U(g)U(g') = c(g,g')U(gg')\,, \quad (18)$$

where $c(g,g')$ is a phase factor—a so-called *cocycle* [14]—then, because of the peculiar form of Eq. (11) the phase factor $c(g,g')$ can be dropped.

The original case of Ref. [1] corresponds to the group of the four Pauli matrices $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$, which is a projective representation of the Abelian dihedral group D_2 of π rotations around three perpendicular axes. Notice that even though the projective representation is non-Abelian (i.e., $\sigma_x\sigma_y = i\sigma_z = -\sigma_y\sigma_x$), the represented group is Abelian ($R_xR_y = R_yR_x = R_z$, R_α denoting a π rotation around the $\alpha = x, y, z$ axis).

The generalization to dimension N in Ref. [1] is again a projective representation of an Abelian group, namely, $\mathbb{Z}_N \times \mathbb{Z}_N$, which is the group of discrete translations on a lattice embedded in a torus. The representation of the group given in Ref. [1] is

$$U(n,m) = \sum_k e^{2\pi ikn/N}|k\rangle\langle k \oplus m|, \qquad (19)$$

which satisfies the composition law

$$U(n,m)U(n',m') = e^{2\pi imn'/N}U(n \oplus n', m \oplus m'), \qquad (20)$$

where $n \oplus n'$ denotes summation mod$N$.

For infinite dimensional Hilbert spaces the POVM related to Eq. (14) generally needs to be expressed in terms of unnormalizable vectors. We rewrite Eq. (14) as follows:

$$|\overrightarrow{\Theta}_g\rangle\!\rangle \equiv U(g) \otimes \mathbb{1} \sum_n |n\rangle \otimes |n\rangle, \qquad g \in \mathbf{G}. \qquad (21)$$

(In general, the variable n may be continuous. In this case the sum would be replaced by an integral.) Moreover, one needs to consider nonmaximally entangled states

$$|\Psi(\lambda)\rangle\!\rangle = \sum_n c_n(\lambda)\,|n\rangle \otimes |n\rangle, \qquad (22)$$

which depend on a physical parameter $\lambda \in [0,1)$ (e.g., this could be a down-conversion gain) such that the state becomes maximally entangled in the limit of $\lambda \to 1$ with $\lim_{\lambda\to 1}|c_{n+1}(\lambda)/c_n(\lambda)| = 1$. Then we introduce the *distortion* operator

$$\mathcal{D}(\lambda) = \sum_n c_n(\lambda)\,|n\rangle\langle n|, \qquad (23)$$

and Eq. (15) now becomes

$$_{1,2}\langle\!\langle \overrightarrow{\Theta}_g | \overleftarrow{\Psi}_{g'}(\lambda)\rangle\!\rangle_{2,3} = U(g')\mathcal{D}(\lambda)U^\dagger(g)\mathcal{T}_{31}, \qquad (24)$$

where

$$|\overleftarrow{\Psi}_g(\lambda)\rangle\!\rangle_{2,3} = \mathbb{1} \otimes U(g)\,|\Psi(\lambda)\rangle\!\rangle_{2,3}. \qquad (25)$$

The teleportation map is achieved for $g' = g$ in the limit of $\lambda \to 1$ as follows:

$$\lim_{\lambda\to 1}\frac{U(g)\mathcal{D}(\lambda)U^\dagger(g)\mathcal{T}_{31}|\phi\rangle_1}{\|U(g)\mathcal{D}(\lambda)U^\dagger(g)\mathcal{T}_{31}|\phi\rangle_1\|} = |\phi\rangle_3. \qquad (26)$$

The continuous variables teleportation of Ref. [3] is an example of infinite dimensional teleportation. The group is the Weyl-Heisenberg group of displacement operators $D(z) = e^{za^\dagger - \bar{z}a}$ (where $[a, a^\dagger] = 1$ for the harmonic oscillator algebra) with composition law $D(z)D(w) = e^{i\,\mathrm{Im}(z\bar{w})}D(z + w)$. Notice that this is just a projective representation of the Abelian group of translations on the complex plane. Equation (13) reads $\int_{\mathbb{C}} \frac{d^2z}{\pi} e^{-|z|^2} = 1$ by taking $|u\rangle = |v\rangle = |0\rangle$ ($|0\rangle$ denoting the vacuum for a). The entangled state is just the down-conversion of the vacuum

$$|\Psi(\lambda)\rangle\!\rangle = \sqrt{1-\lambda^2}\sum_{n=0}^{\infty}\lambda^n|n\rangle \otimes |n\rangle \qquad (27)$$

(a phase factor for λ can always be included into the basis definition). For $\lambda < 1$ one has the teleportation map with distortion

$$\frac{\Omega_z^{(\lambda)}|\phi\rangle_1}{\|\Omega_z^{(\lambda)}|\phi\rangle_1\|} = |\phi_z^{(\lambda)}\rangle_3, \qquad (28)$$

where $\Omega_z^{(\lambda)} = {}_{1,2}\langle\!\langle \overrightarrow{\Theta}_z | \overleftarrow{\Psi}_z(\lambda)\rangle\!\rangle_{2,,3}$, with $|\overleftarrow{\Psi}_z(\lambda)\rangle\!\rangle_{2,3} = \mathbb{1} \otimes D(z)|\Psi(\lambda)\rangle\!\rangle_{2,3}$, and $|\overrightarrow{\Theta}_z\rangle\!\rangle_{1,2} = D(z) \otimes \mathbb{1}|\Theta\rangle\!\rangle_{1,2}$, the latter being the orthogonal POVM corresponding to the eigenvectors of the heterodyne photocurrent [15].

Teleportation for infinite dimensional Hilbert spaces is not restricted to maximally entangled states based on decomposition in Eq. (21). We can define teleportation *filters* that only teleport part of the Hilbert space [16]. An example is the entangled state that results from two harmonic oscillator coherent states $|\alpha\rangle \otimes |\beta\rangle$, through the unitary transformation $U_K = \exp(-i\pi a^\dagger a b^\dagger b)$ where a, b are the annihilation operators. The resulting state is

$$|\Pi\rangle = |\alpha\rangle \otimes |\beta_+\rangle + |-\alpha\rangle \otimes |\beta_-\rangle \qquad (29)$$

$$= |\alpha_+\rangle \otimes |\beta\rangle + |\alpha_-\rangle \otimes |-\beta\rangle, \qquad (30)$$

where $|z_\pm\rangle = |z\rangle \pm |-z\rangle$, which are sometimes called cat states and are parity eigenstates (we have ignored normalization). With this entangled resource/measurement we can only teleport states that lie in the relevant two dimensional parity subspace of the entire Hilbert space.

The universal scheme in the present Letter allows teleportation through entangled measurements based on non-Abelian groups, which has never been considered yet. The simplest case is angular momentum teleportation. We parametrize the group representation matrices as $U(g) = \exp(i\varphi\vec{J}\cdot\vec{n})$, where $\varphi \in [0, 2\pi)$ [17], $\vec{n}$ is a unit vector $|\vec{n}|^2 = 1$ on a sphere, and J_α are customary angular momentum operators. With such a parametrization the invariant measure is $dg = d\vec{n}\sin^2(\varphi/2)d\varphi/8\pi$. The teleportation map is then

$$_{1,2}\langle\!\langle \overrightarrow{\Psi}_{\varphi,\vec{n}}|[|\phi\rangle_1 \otimes |\overleftarrow{\Psi}_{\varphi,\vec{n}}\rangle\!\rangle_{2,3}] = \frac{1}{2J+1}|\phi\rangle_3, \qquad (31)$$

where

$$|\overleftarrow{\Psi}_{\varphi,\vec{n}}\rangle\!\rangle_{2,3} \equiv \mathbb{1} \otimes e^{i\varphi\vec{J}\cdot\vec{n}}|\Psi\rangle\!\rangle_{2,3}, \qquad (32)$$

for a fixed maximally entangled state $|\Psi\rangle\!\rangle$.

In this paper we have presented the essential mathematical description of how entanglement plus local measurement and unitary transformation enables teleportation. In this form we see that teleportation can be described as a

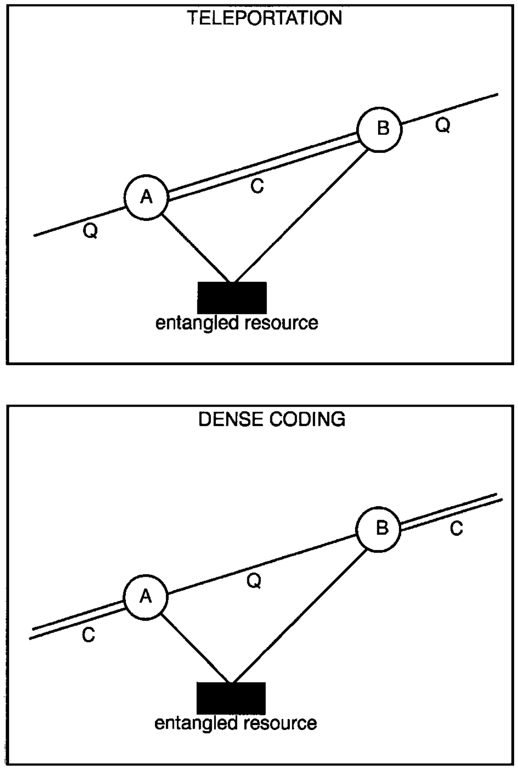

FIG. 1. A schematic representation depicting how both teleportation and dense coding use an entangled resource and classical communication. Time runs vertically and space horizontally. A single line represents a quantum state sent over a quantum (noiseless) channel (Q), a double line represents classical information sent over an ordinary classical communication channel (C). In dense coding the quantum and classical channels are interchanged from that for teleportation. It is already well known that the steps in each protocol converting quantum to classical information (mediated by shared entanglement) involve common Bell state measurements. In this paper, we have furthermore shown that those steps converting classical to quantum information (mediated by shared entanglement) also operate on a common principle: one maximally entangled state may be converted to any other by one-sided (local) operations.

rather special POVM. Dense coding [10] can be given a similar description, however the role played by the classical and quantum information channels is interchanged (see Fig. 1). Both schemes rely on the ability to map shared entanglement to shared entanglement through local unitary transformations. We are thus able to see the common role of local entanglement manipulation in quantum communication protocols.

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