On the Spacing of Planetary Systems

J. Laskar*

Astronomie et Systèmes Dynamiques, IMC, CNRS EP1825, 77 Avenue Denfert-Rochereau, 75014 Paris, France

(Received 9 August 1999)

We present a simplified model of planetary accretion based on conservation of mass, conservation of momentum, and angular-momentum-deficit stability. Within the limitations of this model, we show how the organization of generic planetary systems may be derived from the knowledge of their initial mass distribution. Comparisons with our Solar System and the υ-Andromedae planetary system are presented.

PACS numbers: 95.10.Ce, 05.45.–a, 45.05.+x, 45.70.Qj

Despite significant recent progress in large-scale accretion simulations [1,2], due largely to increased computational power, the formation of the Solar System is far from completely understood [3]. In this paper, we examine a simpler problem. We do not seek to accurately describe the formation of the Solar System. Instead, we use a simplified model for planetary orbital evolution and accretion (in particular, short-period resonances are neglected); but, within the confines of this model, we show that numerical results may be largely recovered through analytical computations. We believe the development of such analytical tools will help to understand more realistic numerical experiments. We also investigate whether the general features of the organization of planetary systems may be derived from simple physical concepts which do not involve the precise mechanisms by which accretion occurs. Technical details and proofs will be presented elsewhere [4].

Since the characteristic time scale for the divergence of nearby orbits in the Solar System is approximately 5 Myr [5,6], the orbital evolution of the planets becomes practically unpredictable after 100 Myr. Thus in the long term, the motion of the Solar System may be described by a random process, where orbits wander erratically in a chaotic zone [7]. In these wanderings, orbits are still constrained by the conservation of energy and angular momentum, whose normal component with respect to the reference plane is $C_n = \sum_{k=1}^{n_p} \Lambda_k \sqrt{1 - e_k^2} \cos i_k$, where e_k are the eccentricities, i_k are the inclinations, a_k are the semimajor axes of the orbits, $\Lambda_k = m_k \sqrt{\mu a_k}$, where $\mu =$ GM., $M.$ and m_k are the solar and planetary masses, G is the gravitational constant, and n_p is the number of planets ($n_p = 8$ in the Solar System, excluding Pluto). In the averaged equations (with respect to the mean longitudes), the quantities Λ_k are constants, which also implies the conservation of the angular momentum deficit (AMD), C, that originates from the eccentricities and inclinations [8]:

$$C = \sum_{k=1}^{n_p} \Lambda_k \left(1 - \sqrt{1 - e_k^2}\, \cos i_k\right). \tag{1}$$

This was used by Laplace [9] to prove that the variations of the planets' eccentricities and inclinations are bounded to first order. The AMD may also be thought of as the amplitude of nonlinearity present in the averaged planetary system. In particular, if the AMD is zero, the averaged motions are planar and circular and stable for all time. However, large values of AMD usually lead to very chaotic behavior [8].

We state that a planetary system is AMD stable if its total AMD is not sufficient to permit planetary collisions. Since this quantity is conserved to all orders [4], AMD stability ensures long-time stability of the averaged system. This is not the case for the complete system, which also contains short-period resonances whose effects are important in the vicinity of collisions. At this stage, we will exclude these short-period resonances and thus consider a simpler model for planetary accretion.

We consider a system comprising a large central body of mass m_0 together with a large number of small bodies $(m_j)_{j=1,n_p}$. When the orbits of two bodies of mass m_1 and m_2 intersect, they may collide and form a new body of mass m_3. We assume that during collision the other bodies are not affected, the mass is conserved ($m_3 = m_1 + m_2$), and the linear momentum in the barycentric reference frame is conserved ($m_3 \dot{\mathbf{u}}_3 = m_1 \dot{\mathbf{u}}_1 + m_2 \dot{\mathbf{u}}_2$). At the time of collision, $\mathbf{u}_3 = \mathbf{u}_1 = \mathbf{u}_2$, so angular momentum is also conserved: $m_3 \mathbf{u}_3 \wedge \dot{\mathbf{u}}_3 = m_1 \mathbf{u}_1 \wedge \dot{\mathbf{u}}_1 + m_2 \mathbf{u}_2 \wedge \dot{\mathbf{u}}_2$. The evolution of orbits during collision is thus completely defined, and the corresponding evolution of the orbital elements may be easily implemented on a computer.

Between collisions, we make the simplifying assumption that the orbits' evolution is similar to their evolution in an averaged system in the presence of chaotic diffusion. More precisely, the orbits evolve in a limited manner (with a random variation of their elements) constrained by the conservation of the total AMD. During a binary collision, the local AMD decreases [4] with a consequent reduction of the total AMD. Collisions cease once the total AMD of the system becomes too small to permit planetary collisions. In what follows, we shall call this model the SPS (simple planetary system) model. The condition for AMD stability is obtained when the orbits of two consecutive planets of semimajor axes a and a' cannot intersect under the assumption that the total AMD of the system has been absorbed by the two planets alone. It can easily be seen that the limit condition of collision is obtained in the planar case, and thus becomes

$$\mathcal{D}(e, e') = \alpha e + e' - 1 + \alpha = 0, \qquad (2)$$

$$C(e, e') = \gamma\sqrt{\alpha}\,(1 - \sqrt{1 - e^2}) + (1 - \sqrt{1 - e'^2}) = C/\Lambda', \qquad (3)$$

where (m, a, e) is the inner orbit, (m', a', e') is the outer orbit, $\gamma = m/m'$, and $\alpha = a/a'$. We seek the minimum value of $C(e, e')$ for which the collision condition (2) is satisfied. Using Lagrange multipliers, we eliminate e' and reduce determination of the limit condition to solving

$$F(e, \alpha, \gamma) = \alpha e + \frac{\gamma e}{\sqrt{\alpha(1 - e^2) + \gamma^2 e^2}} - 1 + \alpha = 0 \qquad (4)$$

in the domain $D_{e,\alpha,\gamma}$, where $e \in [0, 1]$, $\alpha \in [0, 1]$, $\gamma \in [0, +\infty]$. We have $\frac{\partial F}{\partial e}(e, \alpha, \gamma) > 0$ in the domain $D_{e,\alpha,\gamma}$: $F(0, \alpha, \gamma) = -1 + \alpha < 0$; $F(1, \alpha, \gamma) = 2\alpha > 0$; $F(e_0, \alpha, \gamma) = \gamma e_0/\sqrt{\alpha(1 - e_0^2) + \gamma^2 e_0^2} > 0$, with $e_0 = 1/\alpha - 1$. We are thus ensured that the collision equation (4) always has a single solution e_c in the interval $]0, \min(1, e_0)[$. The corresponding value of the AMD $C_c(\alpha, \gamma) = C(e_c, e'_c)$ is then obtained from (3). For any given values of the ratio of semimajor axes α and the ratio of masses γ, we may thus find the critical value $C_c(\alpha, \gamma)$ which permits collision. More precisely, collision is only possible if the total AMD C of the system is larger than $\Lambda' C_c(\alpha, \gamma)$.

In addition, we can show the existence, and compute the limit values, of e_c, e'_c, C_c for $\gamma \to 0$ and $\gamma \to +\infty$. We find $C_c(\alpha, 0) = 1 - 2\sqrt{\alpha(1 - \alpha)}$ for $\alpha < \frac{1}{2}$, $C_c(\alpha, 0) = 0$ for $\alpha \geq \frac{1}{2}$, $C_c(\alpha, +\infty) = 1 - \sqrt{\alpha(2 - \alpha)}$, and $C_c(\alpha, \sqrt{\alpha}) = (1 - \sqrt{\alpha})^2$. We are also able to show that, in the domain $D_{e,\alpha,\gamma}$, $C_c(\alpha, \gamma)$ is increasing with γ and decreasing with α (Fig. 1). Denoting $\delta = 1 - \alpha$, we have also, for $\alpha \to 1$,

$$C_c(\alpha, \infty) \sim C_c(\alpha, 1) \sim \delta^2/2, \quad C_c(\alpha, \sqrt{\alpha}) \sim \delta^2/4. \qquad (5)$$

We now derive the pattern laws obeyed by the planetary distribution resulting from our model of planetary accretion. We begin with an arbitrary distribution of planetesimals with linear mass density $\rho(a)$, and then allow the system to evolve according to the rules described above. We seek the condition under which AMD-stable planetary systems are formed by the random accretion of planetesimals. This condition requires that the final AMD C does not permit planetary crossing among the formed planets.

In the accretion phase, we consider a planetesimal of semimajor axis a and its immediate neighbor, defined as the planetesimal with semimajor axis a' such that there are no planetesimals with semimajor axes in the interval $]a, a'[$. In this case, we may assume that α is close to 1, and, in view of the relations (5), we will use as an approximation of the critical AMD value $C_c(\alpha, \gamma)$,

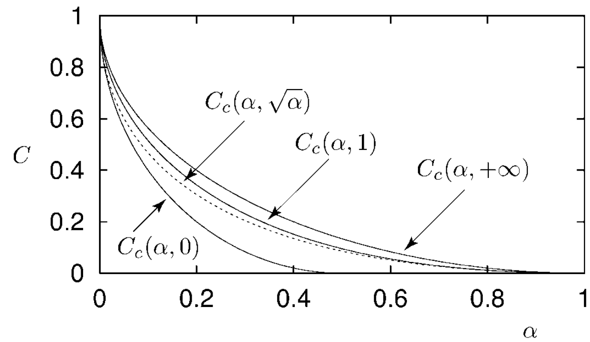

FIG. 1. Values of $C_c(\alpha, \gamma)$ versus α for γ values for which an analytical expression of $C_c(\alpha, \gamma)$ was obtained.

$$C_{c1}(\alpha, \gamma) = k(\gamma)\,(\delta_a/a)^2, \qquad (6)$$

where $\delta_a = a' - a$ and $k(\gamma)$ is a constant. More precisely, since $C_c(\alpha, \gamma)$ is an increasing function of γ, from (5) we have $k(\gamma) = \frac{1}{2}$ for $1 \leq \gamma \leq +\infty$; $\frac{1}{4} \leq k(\gamma) \leq \frac{1}{2}$ for $\sqrt{\alpha} \leq \gamma \leq 1$, and $k(\gamma) \leq \frac{1}{4}$ for $\gamma \leq \sqrt{\alpha}$.

Assuming that the total AMD of the system is C, the mass of the planetesimal of semimajor axis a will continue to increase through accretion as long as $C \geq \Lambda' C_{c1}$. Since a' is the closest neighbor to a, we can assume that all of the planetesimals initially between a and a' were accreted by the two bodies of mass $m(a)$ and $m(a')$. To first order in m/m_0 and δ_a/a, we have $m(a') \sim m(a) \sim \rho(a)\delta_a$. For the limiting case, we then have

$$\frac{\tilde{C}}{\delta_a \rho(a)\sqrt{a}} = k\left(\frac{\delta_a}{a}\right)^2, \qquad (7)$$

where $\tilde{C} = C\sqrt{\mu}$. Equivalently, we can write $\delta_a = (\tilde{C}/k)^{1/3} a^{1/2} \rho^{-1/3}$ and

$$m(a) = (\tilde{C}/k)^{1/3} a^{1/2} \rho^{2/3}. \qquad (8)$$

Using these relations, we may compute the resulting patterns for various initial mass distributions, in particular for $\rho(a) = \zeta a^p$. From Eq. (8), we obtain, for two consecutive planets,

$$\gamma = \frac{m}{m'} = \alpha^{1/2}\left(\frac{\rho(a)}{\rho(a')}\right)^{2/3} = \alpha^{(2p+3)/6}. \qquad (9)$$

From Eq. (6), we thus obtain, for $p = 0$, $\gamma = \sqrt{\alpha}$ and $C_c(\alpha, \gamma) \sim \delta^2/4$ and, for $p \leq -\frac{3}{4}$, $1 \leq \gamma \pm +\infty$ and $C_c(\alpha, \gamma) \sim \delta^2/2$.

Thus, for $p = 0$ [constant distribution, $\rho(a) = \zeta$], we have $k = \frac{1}{4}$, and we obtain $\delta a/\sqrt{a} = (4\tilde{C})^{1/3} \zeta^{-1/3} \delta n$. Since $\delta n = 1$ is the order increment from the planet at a to the adjacent planet at a', integrating this difference relation yields the planetary pattern as follows:

$$\sqrt{a} = \sqrt{a_0} + \left(\frac{\tilde{C}}{2}\right)^{1/3} \zeta^{-1/3} n\,, \qquad (10)$$

The masses are obtained from (8) as $m(n) \sim (2\tilde{C}^2\zeta)^{1/3}n$. The patterns for different values of p are obtained in a similar way (cf. Table I).

We have tested these analytical results on a numerical model of our accretion scheme. The simulation is designed to fulfill the conditions of the SPS model. We start with a large number of bodies of equal mass, with planar orbits having evenly distributed perihelia and following a linear mass density distribution $\rho(a)$. Orbit intersections are monitored step by step.

At each step, the orbits are ordered by increasing a. If an orbit intersects its neighbor, we assume that a collision occurs and the two bodies merge to form a larger body whose orbital parameters are determined by conservation of mass and momentum. This orbit is set aside until the next step, and the succeeding orbit is considered. Between collisions, the orbits do not evolve, except for a random change of their eccentricities under the condition of conservation of total AMD.

The main parameter of these simulations is the final AMD value C_f. Since the AMD decreases during collisions, the total AMD may become smaller than C_f. In this case, the eccentricities are increased by a small amount in order to raise the AMD to the desired final value. This operation is justified because, in a realistic system, close encounters that do not result in collision generally increase the eccentricities and, consequently, also the total AMD [10]. The SPS simulations are extremely fast, as we do not integrate the orbital motions; rather we only determine the intersections of the Keplerian ellipses. This explains why we are able to use 10 000 bodies in the simulations.

It is beyond the scope of this paper to comment extensively on the SPS simulations that we conducted. We present here only the results of SPS simulations corresponding to constant linear distribution [$\rho(a) = \zeta$]. The 10 000 planetesimals are distributed from 0.1 to 10 AU, with a maximum eccentricity of 0.2. The total mass is $m_T = 8 \times 10^{-6}$, and the final AMD is $C_f = 16 \times 10^{-8}$. Units are solar mass and astronomical units (AU); the unit of time is such that $\mu = 1$ (i.e., yr/2π).

As expected, we observe a large variation of the final systems (Fig. 2). For the sake of meaningful statistics, we simulated 5000 different systems, beginning with a random distribution of the perihelia of the initial bodies. In all cases, each system ended up with between four and eight planets. The final values of $\sqrt{a}$ show a wide range (Fig. 2), but as predicted their average values follow a linear distribution with impressive accuracy [Fig. 3(a)]. This is also the case for the mass values, although the agreement is not as perfect [Fig. 3(b)]. Moreover, the theoretical value of the slope obtained from Table I for $\sqrt{a}$ is 0.464, which is also very close to the value 0.468 obtained for six planets

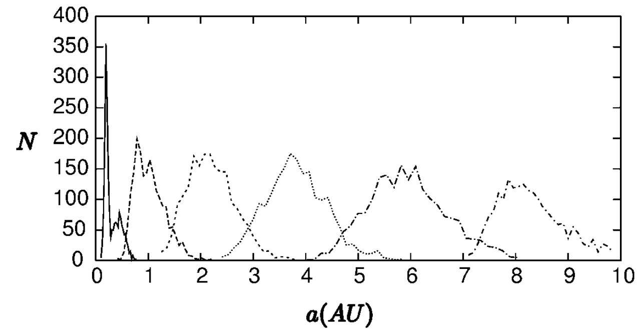

FIG. 2. Distribution of the semimajor axes of the six planet systems: starting with an initial sample of 5000; 1939 systems end up with six planets.

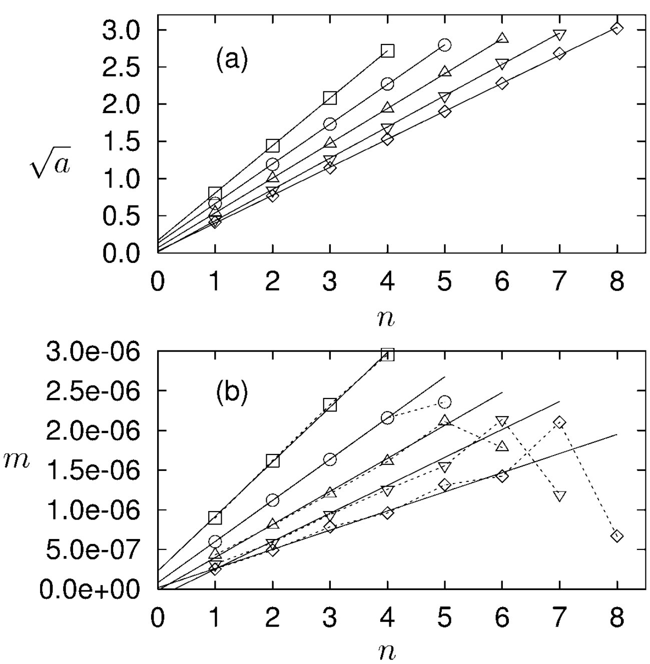

FIG. 3. Average values of $\sqrt{a}$ (a) and m (b) versus the planet index n for the 5000 SPS simulations. The solid lines are linear least-squares fits of the various solutions, depending on the final number n_p ($n_p = 4, 5, 6, 7, 8$) of planets. For the masses (b), the last points are not used for the fit.

TABLE I. Planetary distribution corresponding to different initial mass distribution. For $p = -\frac{1}{2}$, we have $\frac{1}{4} < k < \frac{1}{2}$. $p = 0$ gives a law in n^2 for $a(n)$, while $p = -\frac{3}{2}$ gives a Bodes-like power law.

p	k	a	$m(a)$
0	$\frac{1}{4}$	$\sqrt{a} = \sqrt{a_0} + (\tilde{C}/2\zeta)^{1/3}n$	$(4\tilde{C}\zeta^2)^{1/3}a^{1/2}$
$-\frac{1}{2}$	–	$a^{1/3} = a_0^{1/3} + (\tilde{C}/k\zeta)^{1/3}\frac{n}{3}$	$(\tilde{C}\zeta^2/k)^{1/3}a^{1/6}$
-1	$\frac{1}{2}$	$a^{1/6} = a_0^{1/3} + (2\tilde{C}\zeta)^{1/3}\frac{n}{6}$	$(2\tilde{C}\zeta^2)^{1/3}a^{-1/6}$
$-\frac{3}{2}$	$\frac{1}{2}$	$\log(a) = \log(a_0) + \left(\frac{2\tilde{C}}{\zeta}\right)^{1/3} n$	$(2\tilde{C}\zeta^2)^{1/3}a^{-1/2}$

TABLE II. Slope values for the fitted lines of Figs. 3(a) and 3(b). n_p is the final number of planets and N is the number of cases obtained for a given n_p. Two cases with nine planets are not taken into account.

n_p	4	5	6	7	8
N	451	1965	1939	591	52
$\sqrt{a}$	0.638	0.533	0.468	0.420	0.376
$m(10^{-7})$	6.884	5.189	4.161	3.521	2.415

(Table II) (i.e., for the mean number of planets in the final system). For the masses, we obtain 3.45×10^{-7} instead of 4.17×10^{-7}.

The most famous attempts to set forth laws for the distribution of planetary orbits are surely the Titius-Bode power laws [11,12] which in our study would correspond to an initial mass distribution $\rho(a) = \zeta a^{-3/2}$ (Table I). We prefer here to consider the remark made by Schmidt [13] and also more recently in Refs. [14,15] that, if the Solar System is split into sets of inner and outer planets, the semimajor axes in each set follow a n^2 power law to a high degree of approximation (Fig. 4). In fact, as was recognized in [8], in the framework of this paper it is natural to separate the two systems. Indeed, the AMD of the outer planetary system is large, while the AMD of the inner planets is much smaller. We like to think that the dynamics of the outer planets was initially driven by the total AMD, but then, as their motion became very regular, there was barely any further AMD exchange with the inner planets. The inner planets were then subsequently driven by their own AMD. Using the numerical values for the Solar System [8], we find that the slope of $\sqrt{a}\,(n)$ is 0.14 for the inner planets, while the observed value is 0.199 (cf. Fig. 4); for the outer planets these values are 0.81 and 1.091, respectively.

If we consider the newly discovered planetary system of υ-Andromedae, which has three observed planets [15], the agreement is also striking. The distribution of semimajor axes follows an n^2 law very closely (Fig. 4), suggesting a constant initial density $\rho(a)$. The computed value of the slope of $\sqrt{a}\,(n)$ is 0.56, while the observed value is 0.67. It should be noted here that these results do not depend on the constant but unknown mass factor $\sin i$ (where i is the inclination of the observed planetary system) which cancels in Eq. (10). We consider this agreement to be very encouraging. Indeed the next step will be to take into account the short-period resonances, which will have the effect of increasing the spacing of the planets since they will provide additional possibilities for collisions. It should also be noted that in the present paper, we have not searched for the time scales on which final states are reached, but only for the final states themselves, which are characterized by their final AMD. We expect that the new approach presented here will be helpful in forecasting the results of more realistic numerical simulations, and also in understanding the distribution of orbits in extra-solar planetary systems.

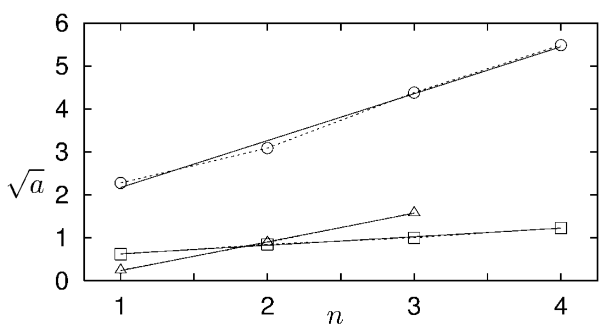

FIG. 4. $\sqrt{a}$ versus n for the inner (squares) and outer planets (circles) of the Solar System, and for the known planets of Upsilon Andromedae (triangles).

Many thanks to Fathi Namouni for extremely helpful discussions. This work was supported by PNP/CNRS and CNUSC.

*Electronic address: laskar@bdl.fr

[1] J. E. Chambers and G. W. Wetherill, Icarus **136**, 304 (1998).
[2] H. F. Levison, J. J. Lissauer, and M. J. Duncan, Astron. J. **116**, 1998 (1998).
[3] J. Lissauer, in *Origins, Proceedings of the ASP 114th Conference* edited by C. E. Woodward, J. M. Shull, H. A. Thronson (Astronomical Society of the Pacific, San Francisco, 1998), p. 327.
[4] J. Laskar (to be published).
[5] J. Laskar, Nature (London) **338**, 237 (1989).
[6] G. J. Sussman and J. Wisdom, Science **257**, 56 (1992).
[7] J. Laskar, Astron. Astrophys. **287**, L9 (1994).
[8] J. Laskar, Astron. Astrophys. **317**, L75 (1997).
[9] P. S. Laplace, *Oeuvres Complètes* (Gauthier-Villars, Paris, 1895), Vol. XI, p. 49.
[10] G. W. Wetherill and G. R. Stewart, Icarus **77**, 330 (1998).
[11] M. M. Nieto, *The Titius-Bode Law of Planetary Distances: its History and Theory* (Pergamon Press, Oxford, 1972).
[12] F. Graner and B. Dubrulle, Astron. Astrophys. **282**, 262 (1994).
[13] O. Schmidt, in *The Origin of the Solar System: Soviet Research, 1925–1991*, edited by A. E. Levin *et al.* (AIP, New York, 1995).
[14] J. C. Pecker and E. Schatzman, *Astrophysique Générale,* (Masson, Paris, 1959).
[15] L. Nottale, G. Shumacher, and J. Gay, Astron. Astrophys. **322**, 1018 (1997).
[16] P. Butler, G. Marcy, D. Fishcher, T. Brown, A. Contos, S. Korzennik, P. Nisenson, and R. Noyes, Astrophys. J. **526**, 916 (1999).