

Quantum Hall Effect at Low Magnetic Fields

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(Received 30 June 1999)

The temperature and scale dependence of resistivities in the standard scaling theory of the integer quantum Hall effect is discussed. It is shown that recent experiments, claiming to observe a discrepancy with the global phase diagram of the quantum Hall effect, are in fact in agreement with the standard theory. The apparent low-field transition observed in the experiments is identified as a crossover due to weak localization and a strong reduction of the conductivity when Landau quantization becomes dominant.

PACS numbers: 73.40.Hm, 71.30.+h, 71.55.Jv, 72.15.Rn

The behavior of the quantum Hall effect (QHE) at low magnetic fields has attracted a lot of attention in recent years, both experimentally and theoretically. Of particular interest has been the fate of the critical states responsible for the transitions between the different quantized Hall plateaus and the form of the global phase diagram of the QHE [1,2]. Recent experiments [3–7] have been interpreted as being incompatible with the global phase diagram [1] based on the levitation of critical states [8,9]. At high magnetic fields, such that the Landau level separation is much larger than the disorder broadening of the Landau bands, these critical states are situated near the centers of the Landau bands. They separate phases with different, quantized values of the Hall conductivity. As the Hall conductivity is constant throughout these phases, the critical energies cannot just terminate when magnetic field or disorder is changed. Either they can move to infinite energy or they can terminate when they intersect another critical state with the opposite change in Hall conductivity. Both of these scenarios are realized: the critical energies at the centers of the Landau bands move to infinite energy as the magnetic field becomes infinitely strong, while the termination of critical states has been observed in lattice models, where due to the band structure, states with negative Hall conductivity exist [10].

Since in the absence of a periodic potential, no states with negative Hall conductivity exist, Khmelnitskii and Laughlin have argued that the existence of critical states at high magnetic fields can only be reconciled with the absence of extended states at zero magnetic field, predicted by the scaling theory of localization in two dimensions [11], if the critical states float above the Fermi energy, when the magnetic field is decreased towards zero. The scaling theory of the integer QHE [12] predicts the Hall conductivity to be quantized at ne^2/h between the critical energies, with integer n , and to be $(n + 1/2)e^2/h$ at the critical energies [2]. The dissipative conductivity vanishes, except at the critical energies, where it takes on values of the order of e^2/h . In Fig. 1 the levitation of the critical states is sketched and the resulting behavior of the conductivities and resistivities is shown. Note that it is not possible to predict the behavior of the resistivities in the

phase with zero Hall conductivity, as their values depend on the way the conductivities tend to zero.

Since the QH plateau transitions are quantum phase transitions, the discussion presented above is concerned with the phase diagram of an infinite system at zero temperature. Experimentally, only finite systems at finite temperature are accessible. In numerical simulations, also the restriction to finite systems applies. When comparing the result of experiments with the predictions of scaling theory, it is therefore imperative to consider the effects of finite temperatures and/or finite system sizes. While this was appreciated in early theoretical and experimental work [8,15], it

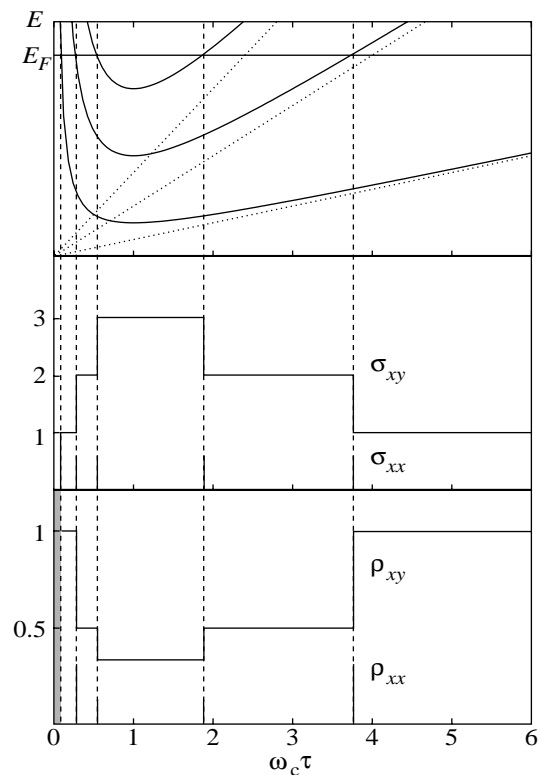


FIG. 1. Sketch of the magnetic field dependence of the critical energies according to the levitation picture ($E_n = (n + 1/2)\hbar\omega_c[1 + 1/(\omega_c\tau)^2]$ [9]) (top), the corresponding (dimensionless) conductivities (center), and resistivities (bottom).

has wandered out of the focus of much of the recent work. It is the purpose of this paper to show that the experiments on the low-field QHE can be understood within the standard scaling theory, obviating the need for a more exotic explanation. We will restrict our attention to the integer QHE and consider interaction effects only on the level of weak localization corrections.

We will start our argument by considering the appropriate starting points for a renormalization of the conductivities, the bare conductivities σ_{ij}^0 , corresponding to short length scales or high temperatures [16]. At high temperatures and low magnetic fields, quantum effects are negligible and the conductivities can be calculated from kinetic equations to give the Drude expressions

$$\sigma_{xx}^0 = \frac{\sigma_0}{1 + (\omega_c \tau)^2}, \quad (1a)$$

$$\sigma_{xy}^0 = \omega_c \tau \sigma_{xx}^0, \quad (1b)$$

with $\sigma_0 = e^2 n_c \tau / m^* = e n_c \mu$, $\omega_c = eB/m^*$, and n_c , $\tau = \ell / v_F$, ℓ , and μ are the carrier density, transport time, elastic mean free path, and the mobility, respectively (Fig. 2). In terms of resistivities, the classical values are $\rho_{xx}^0 = 1/\sigma_0$, independent of the magnetic field B , and $\rho_{xy}^0 = B/en$. Quantum effects modify these results in two ways: quantum interference leads to localization, and Landau quantization drastically modifies the density of states at strong magnetic fields. Quantum mechanically, three energy scales are relevant: the cyclotron energy $E_B = \hbar \omega_c$, the disorder broadening of the Landau bands Γ , and the thermal energy $E_T = k_B T$. The ratio of the former two depends on the strength of the magnetic field and the strength and range of the disorder [17] and corresponds to the classical quantity $\omega_c \tau$, characterizing the classical effects of the magnetic field and disorder in Eqs. (1). While E_B/Γ and $\omega_c \tau$ are not identical, $\omega_c \tau = 1$ can serve as an estimate of the point where Landau level quantization becomes important. The ratio of cyclotron energy to temperature, $E_B/E_T = \hbar e B / m^* k_B T$, determines whether the classical expression for σ_{xx} is appropriate or Landau quantization has to be taken into account. In

the limit of strong magnetic fields, such that Landau level mixing can be neglected, the high temperature (or short length scale) conductivity is qualitatively well described within the self-consistent Born approximation (SCBA) [17]. In this approximation, the conductivity vanishes at zero temperature for integer filling factors $\nu = n_c 2\pi l_c^2$, with the magnetic length $l_c^2 = \hbar / eB$, due to a vanishing density of states. For smooth random potential, relevant to most experiments on GaAs/AlGaAs heterostructures, the peak value of σ_{xx}^0 in SCBA is given by $(l_c^2 / \pi d^2) (e^2 / h)$, independent of the Landau level index. d is the range of the disorder potential (Fig. 2) [18]. Landau level quantization can thus lead to a strong reduction in the conductivity compared to the Drude result, provided that the magnetic field is strong enough, i.e., $\omega_c \tau > 1$.

The localizing effect of quantum interference is most important for the occurrence of the QHE and it is the key to our understanding of the temperature dependence of the resistivities. In the absence of a magnetic field, quantum interference leads to a size-dependent reduction of the conductivity [19]

$$\sigma_{xx}(L) = \sigma_{xx}^0 - \frac{2e^2}{\pi h} \log\left(\frac{L}{\ell}\right). \quad (2)$$

This weak localization expression is valid for large σ_{xx} . While the system size dependence is logarithmically weak for small system sizes, scaling theory predicts that it will eventually lead to complete localization and vanishing conductivity. The corresponding corrections to the Hall conductivity are given by [20]

$$\sigma_{xy}(L) = \sigma_{xy}^0 - \omega_c \tau \frac{4e^2}{\pi h} \log\left(\frac{L}{\ell}\right). \quad (3)$$

Again, the decrease in the Hall conductivity is the precursor of the vanishing Hall conductivity at low fields predicted by scaling theory (Fig. 1). In terms of the resistivities, these corrections lead to a logarithmic increase in the dissipative resistivity, while the Hall resistivity remains unchanged. In addition to the disorder effects, Coulomb interactions lead to logarithmic corrections to σ_{xx} , but not to σ_{xy} [19]. While these effects are important for a detailed comparison with experiment, they do not change the conclusions of the present discussion and will be neglected in the following.

In the presence of a magnetic field, quantum interference effects are reduced and the system size dependence of σ_{xx} becomes even weaker [21],

$$\sigma_{xx}(L) = \sigma_{xx}^0 - \frac{1}{\pi^2 \sigma_{xx}^0} \left(\frac{e^2}{h}\right)^2 \log\left(\frac{L}{l_c}\right). \quad (4)$$

This means that localization effects become strong, when the system size exceeds the localization length

$$\xi^0 = l_c \exp(\pi^2 \sigma_{xx}^0{}^2 \hbar^2 / e^4), \quad (5)$$

defined by $\sigma_{xx}(\xi^0) = 0$. In contrast to the zero field case, the system then does not become completely localized but

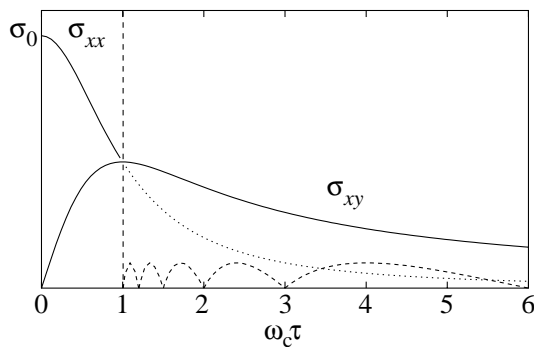


FIG. 2. Conductivities on short length scales: the Drude results (solid lines and dotted line) and the SCBA result for σ_{xx} (dashed lines), appropriate for $\omega_c \tau \gg 1$.

exhibits a series of critical energies at which the conductivity remains finite and the Hall conductivity changes by e^2/h as shown in Fig. 1. The effect of a finite temperature can be incorporated in the present discussion by replacing the system size L by a phase coherence length L_Φ that diverges as the temperature tends to zero.

From scaling theory, the following scenario for the temperature or system size dependence emerges: on small length scales or at high temperatures, classical Drude theory applies. At high magnetic fields, the effects of Landau quantization become visible, when $\pi^2 E_T/E_B \approx 1$ [22]. In GaAs and for $T = 4.2$ K this happens at a magnetic field of about 2 T. At lower temperatures, the Drude expression for σ_{xx}^0 is valid only up to about $\omega_c \tau = 1$ beyond which the SCBA result $\sigma_{xx}^0 \leq e^2/\pi h$ becomes appropriate. Localization effects leading to the QHE become important when the system size and phase coherence length exceed the localization length ξ^0 . If σ_0 exceeds e^2/h , this length scale very rapidly becomes larger than the phase coherence length in present day experiments, provided $\omega_c \tau < 1$. However, around $\omega_c \tau = 1$ the bare conductivity drops below e^2/h and L_Φ can exceed ξ^0 at low temperatures. In particular, near integer filling factors the conductivity is very small and the crossover length ξ^0 becomes small. The point $\omega_c \tau \approx 1$ separates two regions with very different temperature behaviors: at low fields, ρ_{xx} increases slowly with decreasing temperature, while at higher fields, ρ_{xx} decreases, most strongly near integer filling factors, due to the onset of strong localization on the quantum Hall plateau. At the crossover point the resistivity will be only very weakly temperature dependent. Note, however, that this point does not correspond to a critical point in the zero temperature phase diagram. Up to the crossover point near $\omega_c \tau = 1$ deviations from Drude behavior are small so that near the crossing point $\rho_{xx} = \rho_{xy}$.

We thus find that standard scaling theory predicts the essential features of the experiments that have been interpreted as showing a low-field QH-insulator transition: A magnetic field at which ρ_{xx} is temperature independent has been observed in various experiments [5–7,23,24]. This “critical” field separates an “insulating” low-field region with weak temperature dependence from a metallic QH region with stronger temperature dependence on the high field side. At this “transition” Hall and dissipative resistivity are approximately equal [5,7]. The value of the resistivities at this transition is approximately $1/\sigma_0 = 1/en_c\mu$. The density dependence of this value should thus follow the density dependence of the zero-field mobility [5].

It should be stressed that the validity of this argument goes beyond the validity of the employed approximations. The physical mechanism responsible for the drastic change in the temperature dependence near $\omega_c \tau = 1$ is the suppression of the bare conductivity at high fields due to the gaps in the density of states as a result of Landau quantization. This leads to the strong field dependence of the crossover scale ξ^0 . For a more quantitative agreement with

experiment, the bare conductivities should be evaluated in SCBA taking into account Landau level mixing and higher order corrections should be included in Eq. (4).

The question arises, under which conditions the non-monotonic dependence of the Hall conductivity predicted by the levitation scenario could be observed. Khmelnitskii and Laughlin have argued that the plateau transitions are given by the condition that the Drude Hall conductivity σ_{xy}^0 equals half-integer multiples of e^2/h . This implies a lower bound on $n_c\mu$ for the occurrence of the QHE. The maximum value of σ_{xy}^0 is $\sigma_0/2$ at $\omega_c \tau = 1$. Thus, for $\sigma_0 < e^2/h$ there are no plateau transitions and hence no QHE. The reentrant plateau transitions occur for $\omega_c \tau < 1$, where Drude theory is the appropriate expression for the bare conductivity. The minimum σ_0 for the occurrence of the $n = 2$ plateau is $3e^2/h$ and the minimum crossover length ξ^0 at $\omega_c \tau = 1$ is $l_c \exp[(3\pi/2)^2] = 4.4 \times 10^9 l_c$, a macroscopic quantity for magnetic fields in the tesla range. The zero temperature phase diagram with the levitating critical states at low fields is thus of very little importance for experiments on the QHE at low magnetic fields. Even though scaling theory predicts a very different behavior at zero temperature, at all but exponentially low temperatures it predicts a linear increase of the Hall resistivity up to fields where $\omega_c \tau \approx 1$ and the onset of monotonically increasing quantum Hall plateaus beyond.

The system behaves differently, when only the $n = 1$ plateau is observable. The bare conductivity at the low-field QH-insulator transition can then be of the order of $e^2/2h$ and the crossover scale ξ^0 can be microscopic. At this transition scaling behavior should be observable [5,24,25].

From these considerations, we are led to conclude that the recent experimental observation of a temperature independent resistivity at low magnetic fields does, in fact, not contradict the scaling theory of the QHE, but rather is an expected finite-temperature effect. We further see that experiments on the low-field behavior of QH systems reveal only very limited information on the zero-temperature quantum phase transitions. In particular, they do not give much insight into the nature of the insulator phase below the lowest QH transition. The experiments can, however, help to improve our understanding of the finite-size and finite-temperature effects associated with weak localization.

The situation is quite different on the high magnetic field side. Here, the SCBA applies as the starting point for the renormalization of the conductivities [15] and the bare conductivity in the lowest Landau level is less than e^2/h . Thus it is possible to reach the asymptotic scaling regime, both in experiments and in numerical simulations [2]. The results for finite temperatures/system sizes can reliably be extrapolated by finite-size scaling. However, even here the nature of the insulating phase remains quite elusive experimentally. In order to study the insulating phase, it is necessary to go beyond the scaling region of the QH-insulator transition. Numerically, it has been found that

the Hall resistivity remains quantized at h/e^2 throughout the region where scaling behavior is observed [26]. At the high-field end of the scaling region the longitudinal resistivity was found to be up to $16h/e^2$ [27], making accurate measurements of the much smaller Hall resistivity difficult. At zero temperature the width of the scaling region shrinks to zero. The experimentally observed quantization of the Hall resistivity through the transition [28] is thus likely to be a confirmation of scaling behavior and is no indication of the transport properties of the insulating phase at zero temperature.

In conclusion, I have discussed the behavior of the quantum Hall effect at low magnetic fields as expected from the scaling theory of the QHE. The large localization length in a magnetic field in two dimensions restricts the observability of the levitating critical states to exponentially small temperatures and exponentially large systems. At accessible temperatures and system sizes the Hall resistivity will be a monotonically increasing function of magnetic field. Near magnetic fields, such that $\omega_c \tau \approx 1$, the temperature dependence of the dissipative resistivity changes from weakly increasing at low magnetic fields to decreasing at higher magnetic fields, in accordance with recent experiments. At this approximately temperature-independent point ρ_{xx} and ρ_{xy} are of equal magnitude.

I acknowledge stimulating discussions with Z. Wang and X. C. Xie on the quantum Hall insulator and the hospitality of the Institute for Theoretical Physics at Santa Barbara. This work was performed within the research program of the SFB 341 of the DFG and supported by the NSF at ITP.

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