Equilibrium Distribution of Heavy Quarks in Fokker-Planck Dynamics

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We obtain an explicit generalization, within Fokker-Planck dynamics, of Einstein's relation between drag, diffusion, and the equilibrium distribution for a spatially homogeneous system, considering both the transverse and longitudinal diffusion for dimension $n > 1$. We provide a complete characterization of the equilibrium distribution in terms of the drag and diffusion transport coefficients. We apply this analysis to charm quark dynamics in a thermal quark-gluon plasma for the case of collisional equilibration.

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The velocity distribution of objects subject to a (thermal) background plays an important role in a number of scientific fields, including plasma physics and astrophysics [1], nuclear physics [2,3], and, more generally, in kinetic theory [4–6]. The Fokker-Planck equation is a popular tool to study this distribution. It can be motivated in a number of ways. One method is to create a Langevin equation [4,7], which describes the stochastic behavior of a single object propagating with random noise. Another method comes by taking a master equation, such as the linearized Boltzmann-Vlasov equation [6], and performing a Landau soft-scattering approximation. Finding recent theoretical calculations for the transport coefficients of the Fokker-Planck equation based on a microscopic theory [2,3], we recognized the need to establish a simple procedure for understanding the relation between the transport coefficients of the Fokker-Planck equations as determined in such microscopic calculations, and the resulting properties of the equilibrium distribution.

After a brief summary of the recent developments in the Fokker-Planck studies of equilibrating heavy quarks in a quark-gluon plasma, we generalize and apply a well-known relation between kinetic coefficients and the equilibrium distribution of the Fokker-Planck equation in a spatially homogeneous environment [8]. In that way we are able to relate the drag and diffusion coefficients to the shape of the equilibrium distribution. We stress the importance of including both transverse and longitudinal diffusion to maintain a consistent equation. A simple test follows which exactly determines when the equilibrium distribution obeys Boltzmann-Jüttner statistics or the more general Tsallis statistics [9]. We also discuss how to choose the transport coefficients in order to attain the Boltzmann-Jüttner distribution, and address some issues related to the difference between the stopping power and the drag and diffusion coefficients.

The statistical properties of an ensemble of objects (particles) can be expressed in terms of the one-particle distribution function, $f(\vec{x}, \vec{p}, t)$. This density, when multiplied by the $2n$-dimensional phase-space volume element $d^n x d^n p$, gives the probability of finding the object in this infinitesimal region of phase space. We have introduced the dimensionality n explicitly, and we will primarily pursue the physical case $n = 3$, with the case $n = 1$ also of interest due to its exceptional character. We assume that $f(\vec{x}, \vec{p}, t)$ obeys a Boltzmann-Vlasov master equation of the form [6,8]

$$\frac{\partial}{\partial t} f + \dot{\vec{x}} \cdot \nabla_x f + \dot{\vec{p}} \cdot \nabla_p f = \int d^n k \, [W(\vec{p} + \vec{k}, \vec{k}) f(\vec{x}, \vec{p} + \vec{k}, t) - W(\vec{p}, \vec{k}) f(\vec{x}, \vec{p}, t)]; \tag{1}$$

$$f = f(\vec{x}, \vec{p}, t), \qquad \dot{\vec{x}} = \frac{d\vec{x}}{dt} = \frac{\vec{p}}{E}, \qquad \dot{\vec{p}} = \frac{d\vec{p}}{dt} = \vec{F}(\vec{x}). \tag{2}$$

In the nonrelativistic limit, $E \to m$, but otherwise our notation is applicable to both classical and relativistic mechanics. The collision term has two parts: in the first gain term the transition rate $W(\vec{p}_1, \vec{k})$ represents the rate that a particle with momentum $\vec{p}_1 = \vec{p} + \vec{k}$ loses momentum $\vec{k}$ due to reactions with the background. The second term represents loss due to scattering out. The collision term is strictly local, depending only on momenta of particles, but it depends on position $\vec{x}$ indirectly, because W incorporates any background inhomogeneity.

Expanding the gain term about $\vec{p}$ to second order in $\vec{k}$ leads to the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} = \frac{\partial}{\partial p_i} A_i f + \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} B_{ij} f \, , \tag{3}$$

where we are using the Einstein summation convention for repeated indices i and j. We have introduced the transport coefficients of drag and diffusion, respectively:

$$A_i = A_i(\vec{p}) = \int d^n k k_i W(\vec{p},\vec{k}),$$
$$B_{ij} = B_{ij}(\vec{p}) = \frac{1}{2}\int d^n k k_i k_j W(\vec{p},\vec{k}). \tag{4}$$

It is generally believed that the Fokker-Planck equation describes well the approach to (thermal) equilibrium. However, since we shall find that this is not guaranteed, we record yet another independent way to motivate the form of the Fokker-Planck equation (3), the Ito-Langevin method. Consider the Langevin system of equations:

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{E},$$
$$\frac{dp_i}{dt} = F_i(\vec{x}) + G_i(\vec{x},\vec{p}) + D_{ij}(\vec{x},\vec{p})\eta_j(t), \tag{5}$$

where the noise term $\vec{\eta}$ is Gaussian white noise with $\langle \eta_i(t)\rangle = 0$, and $\langle \eta_i(t)\eta_j(t')\rangle = \delta_{ij}\,\delta(t-t')$. Using Ito's formula one shows that this Langevin system corresponds to the Fokker-Planck equation [7]:

$$\frac{\partial f}{\partial t} + \frac{p_i}{E}\frac{\partial f}{\partial x_i} + F_i\frac{\partial f}{\partial p_i} = \frac{-\partial G_i f}{\partial p_i} + \frac{\partial^2(\frac{1}{2}DD^T)_{ij} f}{\partial p_i \partial p_j}. \tag{6}$$

Thus, we see that we can identify $A_i \leftrightarrow -G_i$ and $B_{ij} \leftrightarrow (\frac{1}{2}DD^T)_{ij}$.

While the computation of D_{ij} is not obvious in the Langevin formulation, the master equation approach gives precise formulas. Written in terms of the two body collision reaction matrix elements $\mathcal{M}$, the drag and diffusion, according to Eq. (4), are [2,3]:

$$A_i(\vec{p}) = \frac{1}{2E_p}\int\frac{d^3k}{(2\pi)^3\,2E_k}\int\frac{d^3k'}{(2\pi)^3\,2E_{k'}}\int\frac{d^3p'}{(2\pi)^3\,2E_{p'}} \times \frac{1}{\gamma}\sum|\mathcal{M}|^2(2\pi)^4\delta^4(p+k-p'-k') \times [p_i - p_i']g(\vec{k})\tilde{g}(\vec{k}') \tag{7}$$

$$\equiv \langle\langle p_i - p_i'\rangle\rangle; \tag{8}$$

$$B_{ij}(\vec{p}) = \frac{1}{2}\langle\langle (p_i - p_i')(p_j - p_j')\rangle\rangle. \tag{9}$$

In our case, the incoming particle is different from the background. For each background species, there is a similar additive contribution to the collision integral in Eq. (7). Moreover, as long as the background does not distinguish discrete quantum numbers of incoming particles (such as spin), we can combine these properties in one distribution f. In such a case, we need to average over the initial reaction channels and sum over the open final channels, akin to the evaluation of the cross section, hence the degeneracy factor γ^{-1} of the foreground particle. In Eq. (7) $g(\vec{k})$ is the particle density of the background, assuming that there is a single type of particle, and $\tilde{g}(\vec{k}) = [1 \pm g(\vec{k})]$ represents a Bose enhancement/Pauli suppression factor for scattered background particles, as appropriate. We assume that the background has equilibrated at the temperature T_b.

We are now prepared to study the equilibrium distribution and its relation with the drag and diffusion coefficients. We study the simplest possible case of a spatially homogeneous distribution. In the absence of vectors other than $\vec{p}$ the values of A_i and B_{ij}, which depend functionally on $\vec{p}$ and the background temperature T, must be of the form, where $p^2 = p_i^2$ (summation convention is always implied):

$$A_i(\vec{p},T) = p_i A(p,T), \tag{10}$$

$$B_{ij}(\vec{p},T) = \left(\delta_{ij} - \frac{p_i p_j}{p^2}\right)B_\perp(p,T) + \frac{p_i p_j}{p^2}B_\parallel(p,T). \tag{11}$$

$B_\parallel$ is the longitudinal and $B_\perp$ the transverse diffusion coefficient. In terms of microscopic reaction amplitudes, these three functions are defined by the following expressions [2]:

$$A(p) = \langle\langle 1\rangle\rangle - \frac{\langle\langle \vec{p}\cdot\vec{p}'\rangle\rangle}{p^2}, \tag{12}$$

$$B_\perp(p) = \frac{1}{4}\left[\langle\langle p'^2\rangle\rangle - \frac{\langle\langle(\vec{p}\cdot\vec{p}')^2\rangle\rangle}{p^2}\right], \tag{13}$$

$$B_\parallel(p) = \frac{1}{2}\left[\frac{\langle\langle(\vec{p}\cdot\vec{p}')^2\rangle\rangle}{p^2} - 2\langle\langle\vec{p}\cdot\vec{p}'\rangle\rangle + p^2\langle\langle 1\rangle\rangle\right]. \tag{14}$$

With no external forces and a homogeneous background, Eq. (3) reads

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial p_i}\left(A_i f + \frac{\partial}{\partial p_j}B_{ij} f\right) = -\vec{\nabla}_p\cdot\vec{\mathcal{P}}. \tag{15}$$

A natural requirement for $f_{\rm eq}$, detailed balance, is for the probability current $\vec{\mathcal{P}}$ to vanish [8]:

$$A_i(\vec{p},T) = B_{ij}(\vec{p},T)\frac{\partial\Phi(\vec{p})}{\partial p_j} - \frac{\partial B_{ij}(\vec{p},T)}{\partial p_j}, \tag{16}$$

where we wrote the equilibrium distribution as

$$f_{\rm eq}(p;T,q) = N\exp[-\Phi(p;T,q)]. \tag{17}$$

Here T,q are parameters that may be needed to characterize the shape of the distribution. Graham and Haken [10,11] provide a more general approach which classifies when such a simplification is valid, and how to extend the condition Eq. (16) when it is not valid. For our case, we have verified that Eq. (16) is valid.

Using Eqs. (10) and (11), as well as the fact that in the spatially homogeneous case the equilibrium distribution depends only on $p = |\vec{p}\,|$, Eq. (16) becomes

$$A(p,T) = \frac{1}{p}\frac{d\Phi}{dp}B_{\|}(p,T) - \frac{1}{p}\frac{dB_{\|}}{dp} - \frac{n-1}{p^2}[B_{\|}(p,T) - B_{\perp}(p,T)], \quad (18)$$

which is the desired relation between the shape of equilibrium distribution and the three drag/diffusion coefficients. The special case considered by Einstein arises in the classical problem of a particle that travels through an ideal heat bath and undergoes linear damping (Rayleigh's particle). Substituting the coefficients $A = \gamma$ and $B_{\perp} = B_{\|} = D$ into the relation (18), we obtain the Boltzmann equilibrium distribution Eq. (17) with $\Phi = p^2/2mkT$ only if Einstein's well-known drag-diffusion relation for Brownian motion $\gamma = D/mkT$ is satisfied.

When the equilibrium distribution is known *a priori* and if the diffusion coefficients are also known, then it is an easy matter to use Eq. (18) to find the unique, consistent drag coefficient. However, the reverse is not true for $n > 1$: given the equilibrium distribution and the drag coefficient, there are two diffusion coefficients which must be simultaneously determined, which is in general not possible without a further assumption. One "popular" method to do this is to assume that the tensor B_{ij} is diagonal, that is $B_{\perp} = B_{\|}$ and then to solve the linear first order differential equation to obtain $B_{\|}$:

$$\frac{d}{dp}[e^{-\Phi(p)}B_{\|}(p)] + pA(p)e^{-\Phi(p)} = 0. \quad (19)$$

While discussing the relationships between the drag and diffusion coefficients of the Fokker-Planck equation, we note another interesting relation between these coefficients and the frequently discussed stopping power. The stopping power measures the energy loss per unit distance traveled, and is equal to the energy loss per unit time divided by the particle speed. Thus, in terms of the elementary matrix elements [12]:

$$-\frac{dE}{dx} = \left\langle\!\!\left\langle \frac{E - E'}{v} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \frac{E^2 - E\,E'}{p} \right\rangle\!\!\right\rangle. \quad (20)$$

When combined appropriately with A_i, a relativistically invariant (scalar) quantity is found:

$$p_iA_i + p\frac{dE}{dx} = -\frac{1}{2}\langle\langle (p_\mu - p'_\mu)^2 \rangle\rangle, \quad (21)$$

where we use four-vector notation. Equation (21) shows that the stopping power and the drag coefficient are, in general, two independent quantities. To connect them we need to evaluate also

$$B_{00} \equiv \langle\langle (E - E')^2 \rangle\rangle, \qquad A_0 \equiv \langle\langle E - E' \rangle\rangle. \quad (22)$$

In the nonrelativistic limit these two new quantities are relatively small. The energy loss can be expressed as

$$-\frac{dE}{dx} = \frac{p_iA_i + B_{00} - B_{ii}}{p}, \qquad -\frac{dE}{dt} = A_0. \quad (23)$$

For the Rayleigh particle considered earlier, noting $B_{00} \to 0$, this corresponds to an energy loss:

$$-\frac{dE}{dx} = \gamma\frac{2m}{p}\left(\frac{p^2}{2m} - \frac{3kT}{2}\right) = \gamma m\left(v - \frac{\langle v^2\rangle_T}{v}\right), \quad (24)$$

which vanishes precisely for a thermal velocity.

The problem that we are facing even in the simple spatially homogeneous case is that the Fokker-Planck coefficients cannot simply be chosen to ensure that the "correct" equilibrium distribution results but have already been obtained in terms of elementary collision reaction amplitudes, see Eqs. (12)–(14), and thus the resulting equilibrium distribution is fixed, as can be seen solving and integrating Eq. (18) to obtain Φ. Since the drag and diffusion coefficients are not evaluated exactly but in some valid approximation, typically applying a perturbative expansion, it is more appropriate to analyze the resulting distribution in terms of some useful class. We consider the class of Tsallis statistics [9], which depends on a temperaturelike quantity T and on a parameter q: $\Phi_{\mathrm{Ts}} = \frac{1}{1-q}\ln[1 - (1 - q)E(p)/T]$. The Boltzmann distribution arises when $q \to 1$. Substituting Φ_{Ts} into Eq. (18), we obtain

$$T + (q - 1)E = \frac{dE}{dp}\,\frac{B_{\|}}{pA + \frac{dB_{\|}}{dp} + \frac{n-1}{p}(B_{\|} - B_{\perp})}. \quad (25)$$

Whenever the ratio given by the right-hand side of Eq. (25) becomes linear in E, then Tsallis statistics describe the stationary distribution. When the ratio is constant, then a Boltzmann-Jüttner distribution suffices. We note that for the special case $n = 1$ and nonrelativistic dynamics $dE/dp = v$, Eq. (25) was obtained recently within the Langevin formulation of the Fokker-Planck dynamics [13].

We consider now the drag and diffusion coefficients for a charm quark with mass $m_c = 1.5$ GeV interacting with thermal gluons at $T_b = 500$ MeV calculated using perturbative QCD techniques [2,3]. We have gone to considerable length to ensure that these results apply [14]. Diamonds in Fig. 1 show the ratio Eq. (25). The linear regression fit (straight line) shows that the parameters best describing the distribution as a Tsallis distribution are $q = 1.114$ and $T_{\mathrm{T}} = 135.2$ MeV. The dashed horizontal line in Fig. 1 corresponds to the Boltzmann-Jüttner distribution ($q = 1$ and $T_T = T_b$), which we were expecting to find. The difference to the actual distribution appears to be significant in the wide domain of charmed quark energies studied. Our parametrization of the spectral shape is empirical, yet remarkably accurate, with little statistical improvement arising from further parametrization of the remaining deviation seen in Fig. 1.

The more practical question is what the charmed quark spectrum would actually look like. This is shown in Fig. 2 where a solid line shows the Tsallis distribution as obtained above, compared to Boltzmann-Jüttner shape for

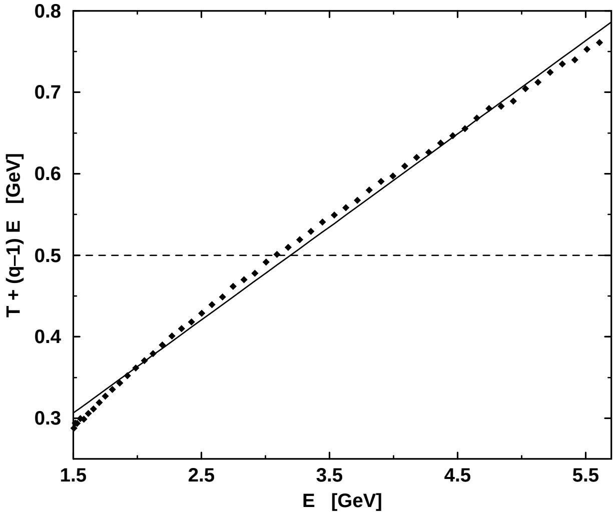

FIG. 1. Calculated data (diamonds) and linear fit for the ratio in Eq. (25) for a charmed quark $m_c = 1.5$ GeV thermalizing in gluon background at $T_b = 500$ MeV. Dashed line: result expected for a Boltzmann-Jüttner distribution, $T = T_b$.

$T_b = 500$ MeV and $m_c = 1.5$ GeV. Assuming that the Tsallis shape would be measured, the spectrum would reveal two components: at low E a "cold" Boltzmann distribution, and for high E a power law with $f_{\mathrm{eq}} \propto E^n$, where in our case $n = \frac{1}{1-q} = -8.8$.

Our study of the equilibrium distribution is systematic, but the results are not definitive, since the properties of transport coefficients are under study. Recent estimates of the stopping power in QGP [15,16], Eq. (23), suggest that glue radiative processes are the dominant contribution to transport coefficients. However, it is not the magnitude of the drag and diffusion transport coefficients (which determine the magnitude of relaxation time towards equilibrium), but the ratio of the transport coefficients which determines the shape of the equilibrium distribution as seen in Eq. (18). Since neither of the three required coefficients has been computed for other than collisional processes, it is not possible to infer the impact of radiative effects on the shape of the equilibrium distribution. However, our methods presented here are easily applied, once these transport coefficients are known.

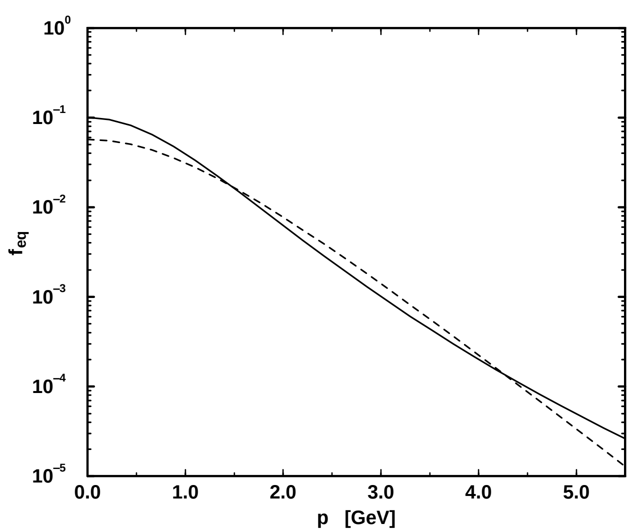

FIG. 2. Normalized equilibrium spectra: Tsallis distribution (solid line) and Boltzmann-Jüttner distribution (dashed line) at $T_b = 500$ MeV, with $m_c = 1.5$ GeV.

In summary, we developed tools which allow us to identify within Fokker-Planck dynamics the equilibrium distribution for given (calculated) drag and diffusion coefficients, or when the stationary distribution is known, to determine the drag or, as a recipe for $n > 1$ dimensions, both longitudinal and transverse diffusion coefficients. We have shown that thermalization of charmed quarks in a quark-gluon plasma by collisional processes on light quarks and gluons leads to a spectral shape well parameterized by the two parameter Tsallis distribution, and we have determined the pertinent spectral parameters for the published microscopic drag/diffusion coefficients.

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