Nonlinear Theory of Nonparaxial Laser Pulse Propagation in Plasma Channels

E. Esarey,[1] C. B. Schroeder,[1,2] B. A. Shadwick,[1,2] J. S. Wurtele,[1,2] and W. P. Leemans[1]
[1]*Center for Beam Physics, Ernest Orlando Lawrence Berkeley National Laboratory, University of California, Berkeley, California 94720*
[2]*Department of Physics, University of California at Berkeley, Berkeley, California 94720*
(Received 21 July 1999)

Nonparaxial propagation of ultrashort, high-power laser pulses in plasma channels is examined. In the adiabatic limit, pulse energy conservation, nonlinear group velocity, damped betatron oscillations, self-steepening, self-phase modulation, and shock formation are analyzed. In the nonadiabatic limit, the coupling of forward Raman scattering (FRS) and the self-modulation instability (SMI) is analyzed and growth rates are derived, including regimes of reduced growth. The SMI is found to dominate FRS in most regimes of interest.

PACS numbers: 52.40.Nk, 41.75.Jv, 52.35.Mw, 52.75.Di

Guiding of intense laser pulses in plasma channels [1] is beneficial to various applications, including harmonic generation [2], x-ray lasers [3], advanced laser-fusion schemes [4], and plasma-based accelerators [5]. A laser pulse in vacuum diffracts after a distance on the order of a Rayleigh length $Z_R = \pi r_0^2/\lambda$, where r_0 is the spot size at focus, $\lambda = 2\pi c/\omega$, and ω is the frequency. A preformed plasma density channel can prevent diffraction, e.g., a channel with a radially parabolic density profile $n(r) = n_0 + \Delta n r^2/r_0^2$ can guide a laser pulse of spot size r_0 provided $\Delta n = \Delta n_c$, where $\Delta n_c = 1/\pi r_e r_0^2$ is the critical channel depth and $r_e = e^2/m_e c^2$ [6]. Plasma channels have been created experimentally by various methods and have been used to guide laser pulses over distances $\lesssim 100 Z_R$ [1,7,8].

Conventional theories of intense, finite-radius pulse propagation in plasmas have assumed the paraxial approximation (PA) [1], which assumes a fixed group velocity and neglects many important finite pulse length effects. In the PA, axial transport of energy within the pulse is not permitted. Hence the PA is incapable of describing many phenomena, e.g., forward Raman scattering (FRS) [9,10], in which intensity modulations arise from an axial transport of energy. The PA does describe the self-modulation instability (SMI) [5,11,12], i.e., intensity modulations from a radial transport of energy. There has been debate within the community [5,9–13] as to which of these instabilities is responsible for intense pulse modulation observed in experiments [13]. A comprehensive theory of FRS and SMI is currently lacking.

In this Letter, a nonlinear theory of nonparaxial pulse propagation is derived that is valid for ultrashort, high-power $P \leq P_c$ pulses in plasmas with or without a parabolic channel. Here $P_c[\mathrm{GW}] = 17(\lambda_p/\lambda)^2$ is the critical power for relativistic self-focusing [1], $\lambda_p = 2\pi c/\omega_p$, and $\omega_p = ck_p = (4\pi n_0 e^2/m_e)^{1/2}$ is the plasma frequency. This theory is first used to analyze pulse propagation in the adiabatic limit, e.g., pulse energy conservation, nonlinear group velocity, damped betatron oscillations, pulse self-steepening, self-phase modulation, and shock formation. In the adiabatic limit the plasma response reduces to a standard third-order nonlinearity in the field. Hence, the adiabatic wave equation typifies a general class of problems in nonlinear media. In the nonadiabatic limit, which includes time dependent coupling to plasma waves, instabilities are analyzed. The explicit coupling and interplay between SMI and FRS are clearly delineated, and analytic expressions for the growth rates are derived, including regimes of reduced growth. The SMI is found to dominate FRS in most regimes of interest.

The wave equation for the transverse component of the normalized vector potential $a_\perp = eA_\perp/m_e c^2$ of the laser field, in terms of the independent variables $\zeta = z - \beta_{g0}ct$ and z, is [1]

$$\left[\nabla_\perp^2 + 2\left(ik + \frac{\partial}{\partial\zeta}\right)\frac{\partial}{\partial z} + \gamma_{g0}^{-2}\frac{\partial^2}{\partial\zeta^2} + \frac{\partial^2}{\partial z^2}\right]\hat{a} = K^2\hat{a}\,, \tag{1}$$

where $a_\perp = (\hat{a}/2)\exp(ikz - i\omega t) + \text{c.c.}$ (c.c. denotes the complex conjugate), ω and k are the central frequency and wave number, $v_{g0} = c\beta_{g0}$ is the linear group velocity of a matched fundamental Gaussian pulse in a channel [14], i.e., $\gamma_{g0}^{-2} = 1 - \beta_{g0}^2 = \omega_p^2/\omega^2 + 4c^2/r_0^2\omega^2$, and $\omega\beta_{g0}/ck = 1$. Here $K^2 \equiv k_p^2(\rho_0 + \delta\rho) - \gamma_{g0}^{-2}\omega^2/c^2$, $\rho_0 = 1 + \Delta n r^2/n_0 r_0^2$, and $\delta\rho$ is the nonlinear plasma response which, in the limits $\hat{a}^2 \ll 1$ and $k_p^2 r_0^2 \gg 1$, is given by [9–12] $(\partial^2/\partial\zeta^2 + k_p^2)\delta\rho \simeq -k_p^2\hat{a}^2/2$, assuming circular polarization such that $a_\perp^2 = \hat{a}^2$.

For a short pulse of length L propagating in a plasma channel, the operators on the left of Eq. (1) scale as $\nabla_\perp \sim 1/r_0$, $\partial/\partial\zeta \sim 1/L$, and $\partial/\partial z \sim 1/Z_R$. In the following analysis, the last two terms on the left of Eq. (1) are small in the parameter regime of interest (underdense plasmas $k_p^2/k^2 \ll 1$) and will be neglected. This is valid provided $|\partial^2\hat{a}/\partial z^2| \ll 2|\partial^2\hat{a}/\partial\zeta\partial z|$, which implies $L \ll 2Z_R$, and $\gamma_{g0}^{-2}|\partial^2\hat{a}/\partial\zeta^2| \ll 2|\partial^2\hat{a}/\partial\zeta\partial z|$, which implies $2L/Z_R \gg (1 + 4/k_p^2 r_0^2)k_p^2/k^2$. These two conditions, along with $k_p^2 r_0^2 \gg 1$, imply $k^2 r_0^2/4 \gg kL \gg k_p^2 r_0^2/4 > 1$. For an underdense plasma $\gamma_{g0}^{-2} \ll 1$, and the $\partial^2/\partial\zeta\partial z$ term

0031-9007/00/84(14)/3081(4)$15.00

dominates. At high densities (e.g., $k_p/k \sim 1$), the $\partial^2/\partial\zeta^2$ term dominates, as in conventional nonlinear optics.

In Eq. (1), the term $2\partial^2/\partial\zeta\partial z$ represents the leading-order correction to the paraxial wave equation. It proves convenient to further approximate this operator by using the paraxial expression for the operator $\partial/\partial z$, i.e., $\partial\hat{a}/\partial z \simeq (-i/2k)(K^2 - \nabla_\perp^2)\hat{a}$. Using this approximation in the term $2\partial^2/\partial\zeta\partial z$, Eq. (1) becomes

$$\left(\nabla_\perp^2 + 2ik\frac{\partial}{\partial z}\right)\hat{a} \simeq \left[K^2 + \frac{i}{k}\frac{\partial}{\partial\zeta}(K^2 - \nabla_\perp^2)\right]\hat{a}\,. \quad (2)$$

The second and third terms on the right represent the lowest order (first order in $1/kL$) contributions of $2\partial^2\hat{a}/\partial\zeta\partial z$.

Equation (2) can be solved using the source-dependent expansion method [1,11], wherein $\hat{a}$ is expanded in a series of Laguerre-Gaussian source-dependent modes, $\hat{a} = \sum_m \hat{a}_m L_m(\chi)\exp[-(1 - i\alpha)\chi/2]$, where $m = 0, 1, 2, \ldots$, $\hat{a}_m(\zeta, z)$ is the complex amplitude, $\chi = 2r^2/r_s^2$, $r_s(\zeta, z)$ is the spot size, $\alpha(\zeta, z)$ is related to the curvature, $L_m(\chi)$ is a Laguerre polynomial of order m, and axisymmetry has been assumed, i.e., $\hat{a} = \hat{a}(r, \zeta, z)$. Assuming that $\hat{a}$ is adequately described by the lowest order mode ($m = 0$), the evolution of the real parameters r_s, α, a_r, and θ, where $\hat{a}_0 = a_r\exp(i\theta)$, is given by

$$\dot{r}_s/r_s = 2\alpha/kr_s^2 - H_I\,, \quad (3)$$

$$\dot{a}_r/a_r = -2\alpha/kr_s^2 + G_I + H_I\,, \quad (4)$$

$$\dot{\alpha} = 2(1 + \alpha^2)/kr_s^2 + 2H_R - 2\alpha H_I\,, \quad (5)$$

$$\dot{\theta} = -2/kr_s^2 - G_R - H_R\,, \quad (6)$$

where $\dot{Q} = \partial Q/\partial z$ (for a function Q), and the subscripts R and I denote the real and imaginary parts. Also, $(G, H) = \sum(G, H)_j$ with $j = a$, b, and c,

$$k^2r_0^2G_a = (2 - \Delta_c r_s^2/r_0^2)(T_A - k) + (1 - \Delta_c r_s^2/r_0^2)T_B\,, \quad (7)$$

$$k^2r_0^2H_a = (\Delta_c r_s^2/r_0^2)(T_A - k) - (1 - 2\Delta_c r_s^2/r_0^2)T_B\,, \quad (8)$$

$$k^2r_0^2G_b = -(1 + \alpha^2)T_A + (i - \alpha)\alpha T_B\,, \quad (9)$$

$$k^2r_s^2H_b = -(1 - i\alpha)^2T_A + (1 + 2\alpha^2 + i\alpha)T_B\,, \quad (10)$$

$$G_c = -\frac{4k_p}{k^2}\int_{\zeta_0}^{\zeta} d\zeta_1\, S(\zeta, \zeta_1)\left[T_C + \frac{r_{s1}^2 T_D}{2(r_s^2 + r_{s1}^2)}\right], \quad (11)$$

$$H_c = -\frac{4k_p}{k^2}\int_{\zeta_0}^{\zeta} d\zeta_1\, \frac{r_s^2 S(\zeta, \zeta_1)}{(r_s^2 + r_{s1}^2)} \times \left[T_C + \frac{r_{s1}^2(r_s^2 - r_{s1}^2)T_D}{2r_s^2(r_s^2 + r_{s1}^2)}\right], \quad (12)$$

where $\Delta_c = \Delta n/\Delta n_c$, $\hat{P} = P/P_c = k_p^2a_r^2r_s^2/16$, $T_A = \theta' - ia_r'/a_r$, $T_B = \alpha' - 2(\alpha + i)r_s'/r_s$, $T_C = k - T_A + 2ia_{r1}'/a_{r1}$, $T_D = -T_B + 4ir_s^2r_{s1}'/r_{s1}^3$, $S = (r_s^2 + r_{s1}^2)^{-1}\hat{P}_1\sin k_p(\zeta - \zeta_1)$, $Q' = \partial Q/\partial\zeta$, $Q_1 = Q(\zeta_1)$, and ζ_0 is chosen before the pulse ($\zeta \leq \zeta_0$). Notice that Eqs. (3) and (4) imply $\partial\hat{P}/\partial z = 2\hat{P}G_I$. When $Q' = 0$, Eqs. (3)–(12) reduce to paraxial limit [1] and $H = G = 0$ describes paraxial vacuum diffraction of a Gaussian beam.

Consider the adiabatic limit in which the pulse length is long compared to the plasma wavelength ($k_p^2L^2 \gg 1$) and coupling to the plasma wave (e.g., FRS) is neglected, i.e., $\delta\rho \simeq -\hat{a}^2/2$. The wave equation then contains a cubic nonlinearity. In this limit, Eqs. (11) and (12) reduce to $k^2r_s^2G_c = 2k^2r_s^2H_c + \hat{P}[\alpha'/2 - (\alpha + 3i)r_s'/r_s]$ and $k^2r_s^2H_c = \hat{P}(\theta' - k - 3ia_r'/a_r)$. This implies $\partial\hat{P}/\partial z + \partial(\delta\beta_g\hat{P})/\partial\zeta = 0$; i.e., the local group velocity is given by $\beta_g \simeq \beta_{g0} + \delta\beta_g(\zeta, z)$, where

$$k^2\delta\beta_g = 2/r_0^2 - (1 + \alpha^2)/r_s^2 - \Delta_c r_s^2/r_0^4 + 3\hat{P}/r_s^2. \quad (13)$$

Furthermore, the total pulse energy $W = \int d\zeta\,\hat{P}$ is conserved, i.e., $\partial W/\partial z = 0$. This is not true for the general nonadiabatic case, since pulse energy is lost to the generation of plasma waves.

In the low power ($\hat{P} \ll 1$) adiabatic limit with $\Delta_c = 1$, $r_s = r_0 + \delta r$, and $\alpha = \delta\alpha$ (where $\delta Q/Q \sim \hat{P}$), we obtain $\delta\beta_g \simeq 3\hat{P}/k^2r_0^2$, and the power evolution is given by $\hat{P} = f(\zeta - 6\hat{P}z/k^2r_0^2)$ where f is a function, e.g., $f(\zeta) = \hat{P}_0\exp(-2\zeta^2/L^2)$ for a Gaussian with a peak power $\hat{P}_0$. This describes self-steepening of the pulse power profile; i.e., the higher the local power, the higher the local group velocity, $\delta\beta_g$, and power is shifted forward within the pulse. The pulse peak moves at a velocity $\beta_{\rm peak} = \beta_{g0} + \delta\beta_{\rm peak}$ with $\delta\beta_{\rm peak} = 6\hat{P}_0/k^2r_0^2$. In the absence of dispersive pulse broadening [from the term $\gamma_{g0}^{-2}\partial^2/\partial\zeta^2$ in Eq. (1)], steepening continues until a shock is formed ($\partial\hat{P}/\partial\zeta \rightarrow \infty$). For a Gaussian $f(\zeta)$, shock formation occurs after a distance $z = Z_S$, where $Z_S = (e^{1/2}/6)kLZ_R/\hat{P}_0$.

Spot size evolution in the low-power adiabatic limit can be examined by perturbing about the zero-power, matched-pulse equilibrium with $\Delta_c = 1$, i.e., $r_s = r_0 + \delta r_s$, $\alpha = \delta\alpha$, $a_r = a_{r0}(\zeta) + \delta a_r$, etc. In particular, Eqs. (3) and (5) imply

$$\left[\left(\frac{\partial}{\partial z} - \frac{2}{kZ_R}\frac{\partial}{\partial\zeta}\right)^2 + \frac{4}{Z_R^2}\right]\frac{a_{r0}\delta r}{r_0} \simeq -\frac{\hat{P}a_{r0}}{Z_R^2}\,. \quad (14)$$

For the initial conditions $\delta r_s = \delta r_0$, $\delta r_s' = 0$, and $\hat{P} = \hat{P}_0\exp(-2\zeta^2/L^2)$,

$$\delta r_s/r_0 = (F_\beta\delta r_0/r_0 + F_\beta^3\hat{P}/4)\cos(k_\beta z) - \hat{P}/4\,, \quad (15)$$

where $F_\beta = \exp(-2z\zeta/Z_\beta L - z^2/Z_\beta^2)$, $k_\beta = 2/Z_R$ is the betatron wave number, and $Z_\beta = kLZ_R/2$ is the betatron damping distance. In the linear limit ($\hat{P} = 0$), Eq. (15) describes damped betatron oscillations of a pulse mismatched ($\delta r_0 \neq 0$) in a channel [14]. Asymptotically, these oscillations damp via $\delta r_s \sim \exp(-z^2/Z_\beta^2)$ for fixed ζ, with a head-tail asymmetry. For finite powers, however, betatron oscillations arise even when $\delta r_0 = 0$, only now with an enhanced damping rate, i.e., $\exp(-3z^2/Z_\beta^2)$. This is the case since a pulse with $\hat{P}_0 > 0$ is no longer matched when $r_s = r_0$ in a channel with $\Delta_c = 1$. Recall that paraxial theory [1] gives a matching condition $r_s^4/r_0^4 = (1 - \hat{P})/\Delta_c$. For $\Delta_c = 1$ and $\hat{P} \ll 1$, this gives $r_s/r_0 \simeq 1 - \hat{P}/4$, precisely the asymptotic ($z \gg Z_\beta$) behavior given by Eq. (15).

Phase distortions (self-phase modulation) also develop. In the limit $\hat{P} \ll 1$ and $\Delta_c = 1$, Eq. (6) implies $\delta\dot{\theta} \simeq (4\delta r/r_0 - 3\hat{P})/kr_0^2$. This results in local frequency shifts via $\delta\omega/\omega = \delta\theta'/k$. Asymptotically, for $z \gg Z_\beta$ (neglecting betatron oscillations), the self-phase modulation due to self-steepening is given by $\delta\dot{\theta} \simeq -4\hat{P}/kr_0^2$, which implies $\delta\omega/\omega \simeq (2/3)\ln[P/P(z = 0)]$.

Numerical solutions to Eqs. (3)–(12) in the adiabatic limit are shown in Figs. 1–3 for the parameters $\lambda = 1\ \mu$m, $r_0 = 10\ \mu$m ($Z_R = 310\ \mu$m), $\lambda_p = 15\ \mu$m ($\Delta n = \Delta n_c = 1.1 \times 10^{18}$ cm^{-3} and $n_0 = 4.9 \times 10^{18}$ cm^{-3}), $a_0 = 0.4$ ($\hat{P}_0 = 0.18$), and $L = 5\ \mu$m (FWHM = 20 fs) with an initially Gaussian profile, $P(0) = P_0 \times \exp(-2\zeta^2/L^2)$. The spot size evolution $r_s(z)$ is shown in Fig. 1 near the (a) front $\zeta = L$, (b) center $\zeta = 0$, and (c) back $\zeta = -L$ of the pulse. The numerical (solid curve) and analytical (dashed curve), Eq. (15), solutions show good agreement in Figs. 1(a) and 1(b). At the back of the pulse, discrepancies arise, e.g., a nonlinear betatron wave number shift; however, excellent agreement is obtained for smaller $\hat{P}_0$. Self-steepening of the power profile $\hat{P}(\zeta)$ is shown in Fig. 2 at $z = 0$ (solid curve), $z = 20Z_R$ (dashed curve), and $z = 40Z_R$ (dotted curve). The velocity of the peak is in good agreement with theory ($\delta\beta_{\mathrm{peak}} = 2.7 \times 10^{-4}$), as is the position of shock formation $Z_s = 0.55Z_\beta/\hat{P} = 48Z_R = 1.5$ cm. The evolution of the intensity profile $a_r^2(\zeta, z)$ is shown in Fig. 3 with the effects of the damped betatron oscillations and self-steepening clearly evident.

A recent paper [15] has proposed using the quasiparaxial approximation (QPA) to analyze the adiabatic limit, in which the $\partial/\partial\zeta$ term in Eq. (1) is replaced by a term proportional to ζ. We note that in the QPA the pulse energy increases via $W \simeq W_0 \exp(z^2/2Z_\beta^2)$, hence, to approximately conserve energy, the QPA is restricted to $z \ll Z_\beta$. Also, we find no evidence for the "enhanced" self-focusing discussed in [15].

Laser-plasma instabilities of finite-radius pulses (as opposed to plane waves) can be examined using the full equations, Eqs. (3)–(12), including coupling to the plasma wave, as in FRS and SMI. Analytically, this is done by expanding Eqs. (3)–(12) about the optically guided, matched-beam equilibrium given by $r_s = r_0$, $a_r = a_0$, $\alpha = 0$, and $\theta' = 0$, where a_0 and r_0 are constants (a flattop axial profile) and $\Delta_c + \hat{P} = 1$ is assumed. Letting $Q = Q_0 + \delta Q$ and $\delta Q = \delta\hat{Q}\exp(ik_p\zeta)$ with $|\partial\delta\hat{Q}/\partial\zeta| \ll |k_p\delta\hat{Q}|$ (modes resonant with the plasma wave) give

$$\mathcal{L}_1\mathcal{L}_2\delta\hat{r} = iC_c\delta\hat{r}\,, \qquad (16)$$

where $\mathcal{L}_1 = \partial^2/\partial\hat{\zeta}\partial\hat{z} + \hat{k}_p\hat{P}$, $\mathcal{L}_2 = (\partial^2/\partial\hat{z}^2 + \hat{k}_\beta^2)\partial/\partial\hat{\zeta} + i\hat{P}$, $C_c = \hat{k}_p\hat{P}^2/2$, $\hat{k}_\beta = k_\beta Z_R = (4 - 2\hat{P})^{1/2}$, $\hat{k}_p = k_p/k$, $\hat{\zeta} = k_p\zeta$, and $\hat{z} = z/Z_R$. Notice that $\mathcal{L}_1\delta\hat{r} = 0$ describes conventional 1D FRS [9,10] and $\mathcal{L}_2\delta\hat{r} = 0$ describes conventional 2D SMI [11,12]. In

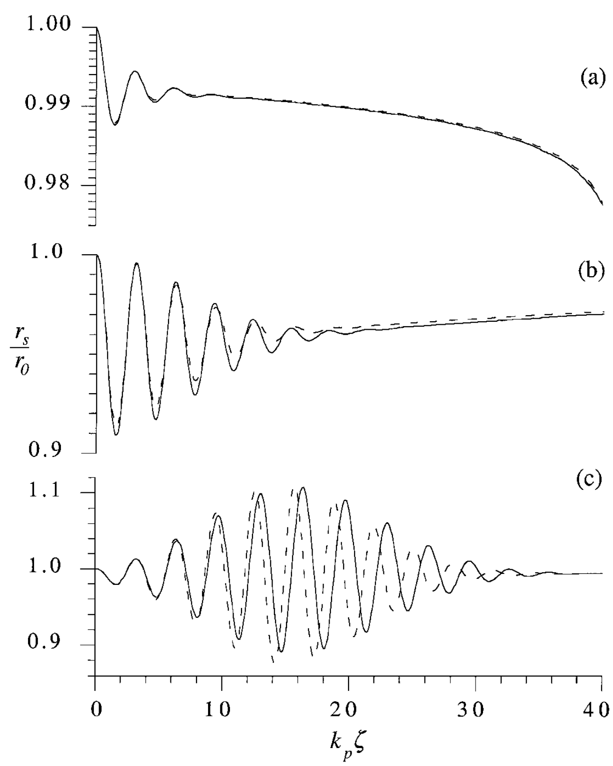

FIG. 1. Spot size $r_s(z)$ at (a) front $\zeta = L$, (b) center $\zeta = 0$, and (c) back $\zeta = -L$ of pulse, from simulation (solid curve) and theory (dashed curve), for $\lambda = 1\ \mu$m, $r_0 = 10\ \mu$m, $\lambda_p = 15\ \mu$m, $\Delta_c = 1$, $\hat{P}_0 = 0.18$, and $L = 5\ \mu$m, with an initially Gaussian profile.

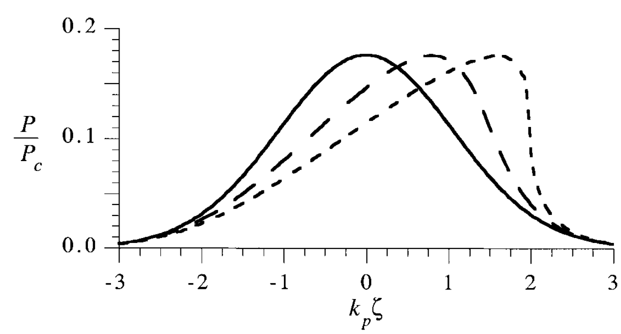

FIG. 2. Power profile $\hat{P}(\zeta)$ at $z = 0$ (solid curve), $z = 20Z_R$ (dashed curve), and $z = 40Z_R$ (dotted curve) for the parameters of Fig. 1.

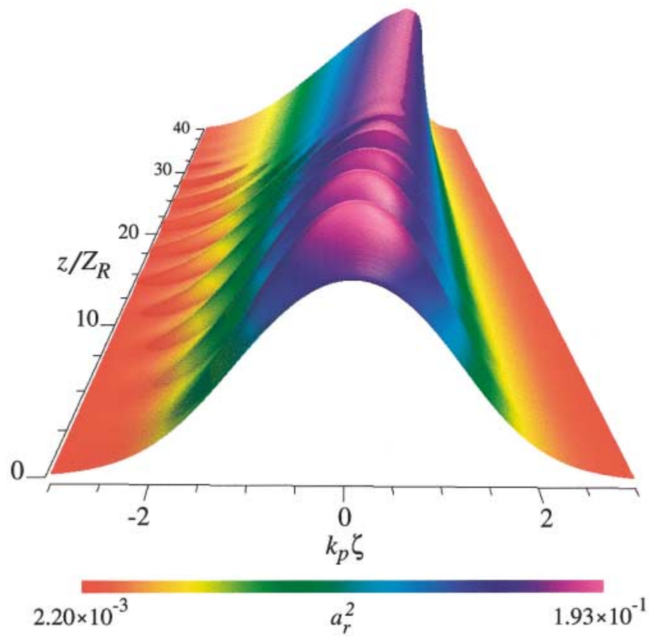

FIG. 3 (color). Intensity profile $a_r^2(\zeta, z)$ for the parameters of Fig. 1.

general, Eq. (16) describes the nonlinear coupling of these two instabilities.

Using Eq. (16), asymptotic expressions for the number of e-folds N_e, $\delta\hat{r} \sim \exp(N_e)$, have been obtained in the appropriate spatial-temporal regimes. Typically, two branches to Eq. (16) are identified, associated with SMI and FRS, with either conventional (C) or reduced (R) growth rates. For the SMI branch, $N_e = (2\hat{P}|\hat{\zeta}|\hat{z}/\hat{k}_\beta)^{1/2}$ (C) is found to be valid in the short-pulse regime $\hat{P}/2\hat{k}_\beta \ll |\hat{\zeta}|/\hat{z} \ll 2\hat{k}_\beta^3/\hat{P}$; $N_e = c_0(\hat{P}|\hat{\zeta}|\hat{z}^2)^{1/3}$ (C) is valid in the intermediate regime $\hat{k}_\beta^3/2\hat{P} \ll |\hat{\zeta}|/\hat{z} \ll 1/2\hat{P}\hat{k}_p^3$, where $c_0 = (1 + i/3^{1/2})3^{3/2}/2^{5/2}$; and $N_e = c_0(\hat{P}|\hat{\zeta}|\hat{z}^2/2)^{1/3}$ (R) is valid in the long-pulse regime $1/\hat{P}\hat{k}_p^3 \ll |\hat{\zeta}|/\hat{z}$. For the FRS branch, $N_e = (4\hat{k}_p\hat{P}|\hat{\zeta}|\hat{z})^{1/2}$ (C) is found to be valid in the short-pulse regime $\hat{k}_p\hat{P} \ll |\hat{\zeta}|/\hat{z} \ll \hat{k}_p\hat{k}_\beta^4/\hat{P}$, $N_e = (2\hat{k}_p\hat{P}|\hat{\zeta}|\hat{z})^{1/2}$ (R) is valid in the intermediate regime $2\hat{k}_p\hat{k}_\beta^4/\hat{P} \ll |\hat{\zeta}|/\hat{z} \ll 2/\hat{P}\hat{k}_p^3$, and $N_e = (4\hat{k}_p\hat{P}|\hat{\zeta}|\hat{z})^{1/2}$ (C) is valid in the long-pulse regime $1/\hat{P}\hat{k}_p^3 \ll |\hat{\zeta}|/\hat{z}$. Note that SMI dominates FRS in the short-pulse (assuming $\hat{k}_\beta\hat{k}_p < 1/2$) and intermediate regimes. FRS dominates SMI in the long-pulse regime; however, here growth is significant only in the tail of long pulses, i.e., $\hat{\zeta} \gg 1/2\hat{k}_p^2\hat{P}$.

As an example, consider parameters relevant to recent experiments on self-modulated laser wakefield acceleration [13]: $\lambda = 1\ \mu$m, $L = 100\ \mu$m (400 fs FWHM), $\lambda_p = 10\ \mu$m ($n_0 \sim 10^{19}$ cm^{-3}), $\Delta_c = 0$, $P \simeq P_c \simeq 2$ TW, and a plasma of length $25Z_R \sim 2$ mm. Near the end of the pulse, $|\zeta| = L$, FRS can occur in the long-pulse regime if $\hat{z} \ll \hat{k}_p^3\hat{P}\hat{\zeta}$ (before transitioning to the intermediate regime at larger z). Letting $\hat{z} = \epsilon^2\hat{k}_p^3\hat{P}\hat{L}$ (with $\epsilon < 1$) gives $N_e \simeq 1.3\epsilon$, i.e., FRS will not undergo significant growth. On the other hand, near the front of the pulse $|\zeta| = L/4$, SMI will reach saturation in the intermediate regime, e.g., $N_e \simeq 12$ after $z = 5Z_R$.

In summary, a nonlinear theory of finite-radius pulse propagation has been developed that includes finite pulse length and group velocity effects. In the adiabatic limit, effects such as the nonlinear group velocity, damped betatron oscillations, and self-steepening were analyzed. In the nonadiabatic limit, the nonlinear coupling of FRS and SMI was described and asymptotic growth rates were derived in various regimes. For sub-ps pulses, SMI dominates in typical regimes. The validity of this theory has been restricted to underdense plasmas ($k_p/k \ll 1$) with $z < Z_S$, but these constraints can be relaxed by a straightforward extension of this theory to include the $\gamma_{g0}^{-2}\partial^2/\partial\zeta^2$ term in Eq. (1).

This work was supported by the Department of Energy under Contract No. DE-AC-03-76SF0098. The authors acknowledge useful conversations with W. B. Mori.

[1] For a review see E. Esarey *et al.*, IEEE J. Quantum Electron. **33**, 1879 (1997).
[2] H. M. Milchberg *et al.*, Phys. Rev. Lett. **75**, 2494 (1995); A. Rundquist *et al.*, Science **280**, 1412 (1998).
[3] D. C. Eder *et al.*, Phys. Plasmas **1**, 1744 (1994); B. R. Benware *et al.*, Phys. Rev. Lett. **81**, 5804 (1998).
[4] M. Tabak *et al.*, Phys. Plasmas **1**, 1626 (1994); S. C. Wilks and W. L. Kruer, IEEE J. Quantum Electron. **33**, 1954 (1997).
[5] For a review see E. Esarey *et al.*, IEEE Trans. Plasma Sci. **24**, 252 (1996).
[6] P. Sprangle *et al.*, Phys. Rev. Lett. **69**, 2200 (1992).
[7] S. P. Nikitin *et al.*, Phys. Rev. E **59**, R3839 (1999); D. Kaganovich *et al.*, Phys. Rev. E **59**, R4769 (1999); P. Volfbeyn *et al.*, Phys. Plasmas **6**, 2269 (1999).
[8] K. Krushelnick *et al.*, Phys. Rev. Lett. **78**, 4047 (1997); S. Y. Chen *et al.*, Phys. Rev. Lett. **80**, 2610 (1998); J. Fuchs *et al.*, Phys. Rev. Lett. **80**, 1658 (1998); C. E. Clayton *et al.*, Phys. Rev. Lett. **81**, 100 (1998).
[9] T. M. Antonsen and P. Mora, Phys. Fluids B **5**, 1440 (1993); P. Mora and T. M. Antonsen, Phys. Plasmas **4**, 217 (1997).
[10] W. B. Mori *et al.*, Phys. Rev. Lett. **72**, 1482 (1994); C. D. Decker *et al.*, Phys. Plasmas **3**, 1360 (1996); W. B. Mori, IEEE J. Quantum Electron. **33**, 1942 (1997).
[11] E. Esarey *et al.*, Phys. Rev. Lett. **72**, 2887 (1994); P. Sprangle *et al.*, Phys. Rev. Lett. **73**, 3544 (1994).
[12] N. E. Andreev *et al.*, Phys. Plasmas **2**, 2573 (1995); N. E. Andreev *et al.*, IEEE Trans. Plasma Sci. **24**, 363 (1996).
[13] R. Wagner *et al.*, Phys. Rev. Lett. **78**, 3125 (1997); C. I. Moore *et al.*, Phys. Rev. Lett. **79**, 3909 (1997); D. Gordon *et al.*, Phys. Rev. Lett. **80**, 2133 (1998).
[14] E. Esarey and W. P. Leemans, Phys. Rev. E **59**, 1082 (1999).
[15] P. Sprangle *et al.*, Phys. Rev. Lett. **82**, 1173 (1999); Phys. Rev. E **59**, 3614 (1999).