Stable Static Localized Structures in One Dimension

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We study the existence, the stability properties, and the bifurcation structure of static localized solutions in one dimension, near the robust existence of stable fronts between homogeneous solutions and periodic patterns.

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The study of static localized structures has attracted a great deal of attention, at least since the time one expected to use magnetic bubbles as main storage elements (see, e.g., [1]). Similar localized structures arise in liquid crystals [2], in gas discharge systems [3,4], and in chemistry [5–7] so that, all together, one has both variational and nonvariational examples. More recently, there has been a new surge of interest in the context of optics, where such structures have again been envisioned as storage elements [8]. For quite some time, it has been recognized that the existence of localized structures does not require bistability, by which is usually meant the simultaneous existence of two stable homogeneous states [6]. Here we will use the theory of dynamical systems to formulate a unified analysis of a large class of stable localized solutions when the dimension of the medium is one. Our theory can be tested as we make predictions on the phenomenology one can observe in the neighborhood of the parameter region where some stationary fronts exist. We will build on the fact that, in the case when a stable homogeneous solution coexists with a stable periodic pattern, Pomeau has described the mechanism of robust existence of stable stationary fronts between the two states [9]: there is a region $\mathcal{F}$ in parameter space whose boundaries correspond to unpinning transitions, where such fronts exist.

We will report on the fact that the region $\mathcal{F}$ is shadowed by a region of existence of stable localized structures when the dimension of the medium is one, as well as in two dimensions when the periodic pattern has compact elementary cells such as hexagons. We consider systems described by variational or nonvariational equations such as

$$\partial_t u = -\frac{\partial V}{\partial u} - v + D_u \nabla^2 u , \qquad \partial_t v = -\gamma v + cu + D_v \nabla^2 v , \tag{1}$$

or

$$\partial_t u = -\frac{\partial V}{\partial u} - \nu \nabla^2 u - \nabla^4 u , \tag{2}$$

where $V = -\mu u^2/2 + u^4/4 - \eta u$. Equation (1) describes a chemical reaction [6], while Eq. (2) is a generalization of the Swift-Hohenberg model [10], which appears in a variety of contexts. We numerically observe localized structures in $\mathcal{F}$ both when the dimension of the medium is one and when it is two (see Figs. 1 and 5). When approaching the unpinning transition, where the front loses its stationarity by a periodic nucleation process which destroys elementary cells at the interface [11,12], the minimal number of cells that one can observe in a localized structure diverges. When crossing the other unpinning transition, where the front loses its stationarity by a periodic nucleation process which creates new elementary cells at the interface, one continues to observe localized structures, but now the maximal number of cells decreases as the parameter departs from the transition. In

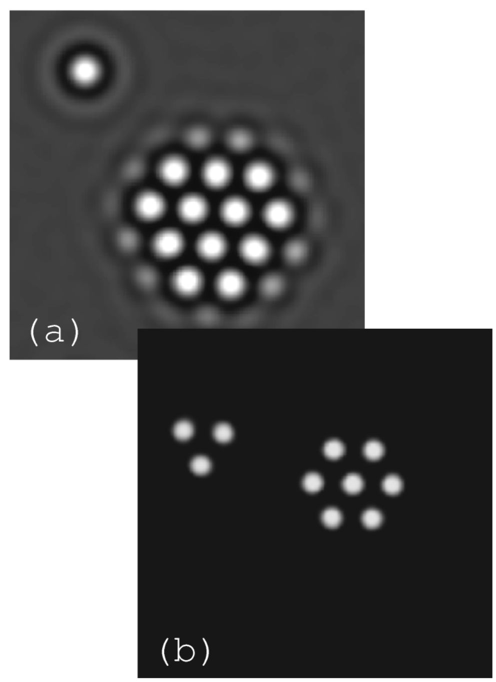

FIG. 1. Localized structures obtained in two-dimensional simulations of Eqs. (1) and (2). In (a), the single cell is stable, but the larger structure is growing. In (b), one observes two-cell and three-cell solutions while the one-cell solutions cannot be stabilized.

the case where the dimension of the medium is one, we will offer a description of the underlying mechanism in terms of the qualitative theory of differential equations.

In the case where the dimension of the medium is one, the second member of Eqs. (1) or (2), which describes the stationary solutions, can be rewritten as a four-dimensional vector field [13] (we assume this dimension to be $2N \geq 4$ in the rest of the discussion) that inherits from the space isotropy the reversibility property [14,15] which is the essential ingredient in the analysis to follow. An orbit of the vector fields does not need to correspond to a stable solution of the partial differential equation (PDE). For reversible critical points (which correspond to parity invariant homogeneous states) and reversible periodic solutions (which correspond to parity invariant periodic patterns), dynamical stability generically implies there is no purely imaginary eigenvalue or purely imaginary Floquet exponent ik_0, $k_0 \neq 0$, because otherwise the corresponding zero growth rate at k_0 would generically induce positive and negative growth rates at nearby wavelengths [dynamically unstable localized solutions of Eq. (2) have been studied in [16] and references therein]. Natural coordinates for the phase space are (u, u_x, v, v_x) for Eq. (1) and $(u, u_x, u_{xx}, u_{xxx})$ for Eq. (2). By definition [14], the phase portrait of a *reversible system* in dimension $2N$ is invariant under the reflection symmetry R about a N-dimensional plane Π. More precisely, for any point X, the forward trajectory of X remains the image under R of the backward trajectory of the point $R(X)$. Orbits invariant under R are called *reversible*. Combining the previous stability argument with the reversibility property implies that all stable and unstable manifolds of reversible critical points and periodic orbits which are dynamically stable have dimension N. In the above cases where the fields components are scalars, Π corresponds to the odd derivatives set equal to zero, so that all of the critical points, which correspond to the stationary homogeneous solutions of the PDE, are reversible. A (stationary) localized solution of the PDE corresponds to a homoclinic curve biasymptotic to a critical point of the vector field: if the homoclinic curve is reversible, it is robust. Reversible periodic orbits are characterized by the fact that they intersect Π at exactly two points. A fundamental result pertaining to these periodic orbits is that they arise in one-parameter families, with the period of the orbit generically varying along the family [14]. The heteroclinic orbit corresponding to a Pomeau front selects one of these periods when a parameter λ varies (for instance, η in the examples given above). Hence, the existence of the one-parameter family of reversible periodic orbits explains the robustness of the stationary front (a similar argument has been used to establish the boundary-induced wavelength selection in [13]). This can be captured on a Poincaré map on a $(2N - 1)$-dimensional section to the family of periodic orbits containing Π, as illustrated in Fig. 2 in the case when $N = 2$. In this section, the family appears as a line P of fixed points P_λ, so that the collection of the stable and unstable manifolds form n-dimensional surfaces $W^s(P)$ and $W^u(P)$ symmetrical with respect to Π, the invariant manifolds of the critical point A appear as $(N - 1)$-dimensional surfaces $W^s(A)$ and $W^u(A)$, so that $W^u(A)$ intersects $W^s(P)$ transversely. When λ varies, the selected fixed point P_λ changes, and the relative positions of $W^u(A)$ and $W^s(P)$, and of $W^u(A)$ and Π, change (as illustrated in Fig. 3), where the boundaries of $\mathcal{F}$ correspond to Figs. 3a and 3c. As illustrated in Fig. 3b, between the boundaries of $\mathcal{F}$, there are transversal intersections of $W^u(A)$ with $W^s(P)$ (for an early relationship

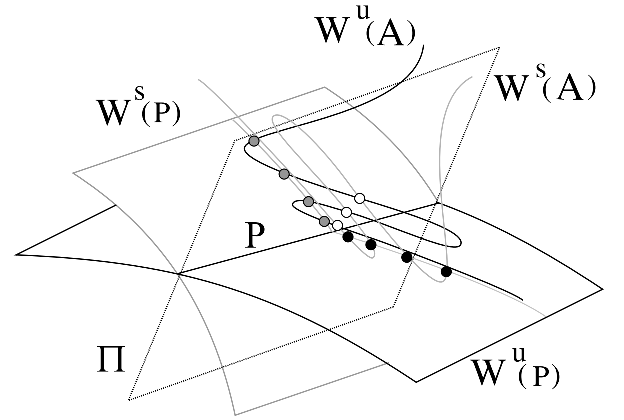

FIG. 2. Phase portrait for a Poincaré section near a family of reversible fixed points when there exists a Pomeau front.

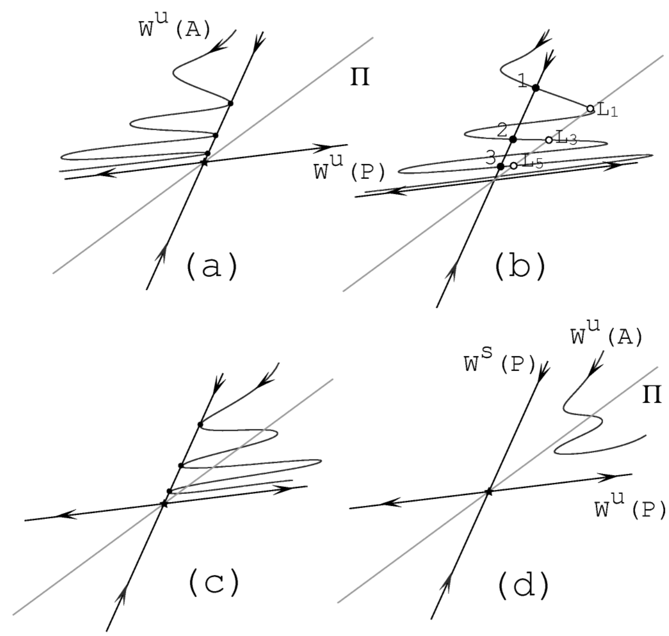

FIG. 3. The evolution of the heteroclinic and homoclinic point structures as the parameter λ is varied. In (a), the stationary front appears. In (b) the existence of the front induces the existence of localized structures. In (c), the stationary front disappears. In (d), localized structure subsists after the stationary front has disappeared.

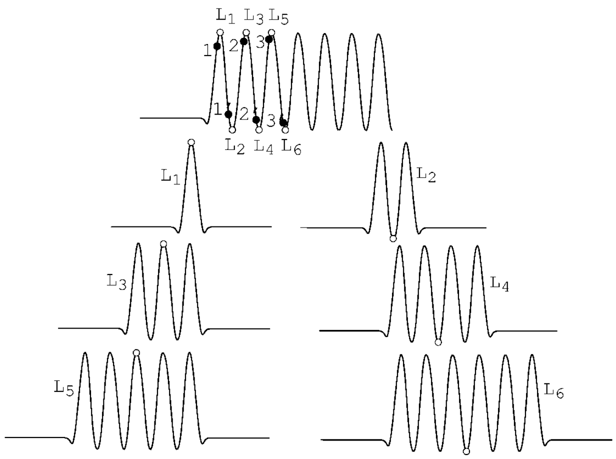

FIG. 4. Correspondence between elementary cells on the structured side of a front and elementary cells of localized structures with an even or odd number of elementary cells, corresponding to the pairing of marked points in Figs. 2 and 3b.

between pinning and transversality, see, e.g., Ref. [17]). This implies transversal intersections of $W^u(A)$ and Π, which by reversibility are also transversal intersections of $W^s(A)$ and Π, thus correspond to homoclinic orbits biasymptotic to A and thereby to localized solutions of the PDE. These solutions arise in pairs, being created and destroyed by saddle-node bifurcations; using the continuity of the spectrum and the stability of the structured solutions, this indicates that the localized solutions of the PDE arise in stable-unstable pairs. Figure 3d illustrates the fact that such homoclinic intersections subsist beyond one boundary of $\mathcal{F}$, until they successively disappear by saddle-node bifurcations. There is a natural pairing between heteroclinic and homoclinic points, illustrated in Figs. 2 and 3c: this corresponds to a pairing between elementary cells that appears on the structured side of a front and in localized solutions, as shown in Fig. 4. The localized solutions appear there in two families, depending on the parity of the number of elementary cells they contain: the homoclinic orbits corresponding to the odd family was recently described in [18] in the case of a Hamiltonian vector field. These two groups correspond to the homoclinic intersections appearing, respectively, in the sections near the two intersection lines P_λ and Q_λ of Π with the families of periodic orbits: the saddle nodes creating and destroying these pairs accumulate in intertwined geometric series (one series for even numbers of elementary cells and one series for odd numbers) which accumulate on the boundaries of $\mathcal{F}$. The ratio of these series is the modulus of the Floquet multiplier of the periodic orbit involved in the front at the boundary of $\mathcal{F}$, which is out on the unit disk and closest to the unit circle. The global bifurcation structure for observable solutions of the PDE is summarized in Fig. 5, where the $(x\text{-}t)$ diagrams should be self-explanatory. The vertical lines between the boundaries of $\mathcal{F}$, and to its right, correspond,

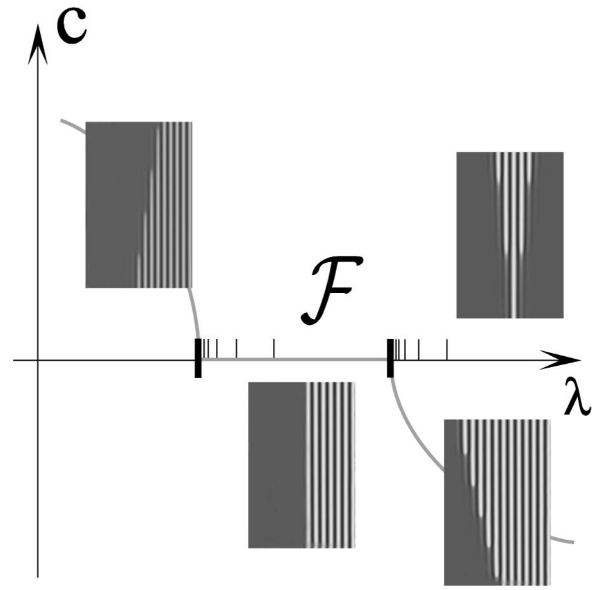

FIG. 5. The bifurcation structure near the pinning region. The horizontal axis is the parameter λ. The vertical axis is the mean velocity of the front. The square root shapes at the onset of nonzero speeds that bound $\mathcal{F}$ are a direct consequence of the saddle-node bifurcation of heteroclinic orbits in Figs. 3a and 3c. Vertical bars correspond to saddle nodes of homoclinic orbits. In the x-t diagrams, the t axis is upward.

respectively, to creation and destruction saddle nodes of stable-unstable pairs of localized solutions.

The geometrical discussion we have offered elucidates the existence, the stability properties, and the bifurcation structure of localized solutions when the dimension of the medium is one, near the robust existence of stable fronts between homogeneous solutions and periodic patterns. One outcome is that in the domain that we have precisely identified, and larger than the Pomeau domain of existence of stable fronts, the periodic pattern can be understood as a juxtaposition of elementary units which are the localized structures. A heuristic analysis *a la* Melnikov, near a nonrobust heteroclinic connection between critical points with purely complex spectra, also leads to the existence of stationary localized structures, however, with a much smaller parameter range of stability [19]. As mentioned previously, the phenomenology in and near $\mathcal{F}$ is observed to be quite comparable when the dimension of the medium is two, with more richness associated with bigger subgroups of the rotation group, but this case still eludes our geometrical approach, despite work in progress. Details of this work are reported in Ref. [20].

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