Optimal Quantum Cloning via Stimulated Emission

Christoph Simon, Gregor Weihs, and Anton Zeilinger
Institut für Experimentalphysik, Universität Wien, Boltzmanngasse 5, A-1090 Wien, Austria
(Received 24 March 1999; revised manuscript received 13 August 1999)

We show that optimal universal quantum cloning can be realized via stimulated emission. Universality of the cloning procedure is achieved by choosing systems that have appropriate symmetries. We first discuss a scheme based on stimulated emission in certain three-level systems, e.g., atoms in a cavity. Then we present a way of realizing optimal universal cloning based on stimulated parametric down-conversion. This scheme also implements the optimal universal NOT operation.

PACS numbers: 03.67.Lx, 03.65.Bz, 32.80.Qk

It is not possible to construct a device that produces an exact copy of an arbitrary quantum system [1]. This impossibility has deep roots. It can be seen as a consequence of the linearity of quantum mechanics. It also prevents the use of EPR correlations for superluminal signaling [2,3]. Nonperfect copying, or cloning, though, is possible. Since the seminal paper of Bužek and Hillery [4], quantum cloning has been extensively studied theoretically. Upper bounds for the possible fidelity of quantum cloners were derived [5], and optimal universal quantum cloning transformations were discovered [6].

All devices proposed so far consist of several quantum gates. This means that it will probably take some time until their practical realization. On the other hand, cloning was originally discussed in the context of stimulated emission [1,3]. It was realized that *perfect* copying is prevented by the unavoidable presence of spontaneous emission [7]. The question arises whether *optimal* cloning (for which the fidelity of the clones saturates the above-mentioned bounds) can be realized with stimulated emission. In this Letter, suggesting realistic scenarios, we show that the answer is "yes."

The cloning procedure will clearly be universal, i.e., equally good for all possible input states, if the cloning system is symmetric under general unitary transformations of the system to be cloned. To be more specific, consider cloning of a general qubit represented by the polarization state of a photon. This requires a population inverted medium whose initial state and whose interaction Hamiltonian with the electromagnetic field are both invariant under general polarization transformations so that it can emit photons of any polarization with the same probability. If a photon interacts with such a medium, it stimulates the emission of photons of the same polarization.

Of course, there is also spontaneous emission of photons of the wrong polarization. The presence of this spontaneous emission is unavoidable because for a given transition the total emission amplitude (i.e., stimulated and spontaneous) is $\sqrt{n+1}$ times the amplitude for spontaneous emission, where n is the number of stimulating photons present. As we want stimulated emission to be possible for all photon polarizations in order to achieve universality, this means that spontaneous emission will also occur for all polarizations.

The photons in the final state can be considered as clones of the original incoming photon. The fidelity of the clones is limited by the presence of the spontaneously emitted photons. The fidelity is defined as $\langle\psi|\rho^{\text{out}}|\psi\rangle$, where $|\psi\rangle$ is the state of the original qubit and ρ^{out} is the reduced density matrix of one of the clones. This is equivalent to the relative frequency of photons of the right polarization in the final state. Starting from one qubit, an optimal universal symmetrical cloner [5] produces M identical clones with a fidelity $F_{\text{opt}}(M) = \frac{2}{3} + \frac{1}{3M}$. Note that $M = 2$ which gives $F_{\text{opt}} = 5/6$ means that there is just one additional qubit besides the original.

In the following we present two possible schemes for the practical realization of quantum cloning. The first one, that uses certain three-level systems as the inverted medium, is optimal in the short-time limit. The second one, based on stimulated parametric down-conversion, is optimal for all times.

We now discuss our procedure for cloning with three-level systems. These systems have a ground level g and two degenerate upper levels e_1 and e_2, connected by two orthogonal modes of the electromagnetic field, a_1 and a_2 (see Fig. 1). The field modes define the Hilbert space of our qubits, i.e., we want to clone general superposition states $(\alpha a_1^\dagger + \beta a_2^\dagger)|0\rangle$. Note that we are talking about photons and polarization in order to be specific, but one is free to think of other systems and other degrees of freedom, as long as they are described by the same formalism. In the interaction picture, the Hamiltonian has the following form:

$$H = \sum_{K=1}^{N} \gamma(\sigma_{+1}^K a_1 + \sigma_{+2}^K a_2) + \text{H.c.}, \tag{1}$$

where $\sigma_{+1(2)} = |e_{1(2)}\rangle\langle g|$ and the index K refers to the Kth atom.

The Hamiltonian (1) is invariant under simultaneous unitary transformations of the vectors $(a_1^\dagger, a_2^\dagger)$ and $(|e_1\rangle, |e_2\rangle)$.

0031-9007/00/84(13)/2993(4)$15.00

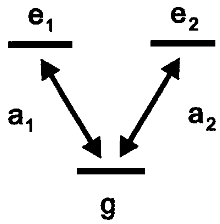

FIG. 1. One possible level structure of systems used for universal cloning, optimal for short interaction times.

Furthermore, we require each atom to be initially in the mixed state

$$\rho_i = \frac{1}{2}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|), \qquad (2)$$

which is invariant under the same unitary transformations. The invariance of both Hamiltonian and initial state together ensures the universality of the cloning procedure. Therefore it is sufficient to analyze the performance of the cloner for one arbitrary incoming one-photon state; we choose $|\psi_i\rangle = a_1^\dagger|0\rangle$. It is interesting to note that in the present scheme the atoms not only act as a photon source but they also play the role of ancillas for the cloning procedure (cf. [4,6]).

We performed numerical computations for systems of a few (up to $N = 6$) atoms. From (1), the time development operator $U = e^{-iHt}$ for the whole atoms-photons system was calculated. Use was made of the fact that N_1 and N_2, which denote the sum of the number of photons plus the number of excited atoms for modes 1 and 2, respectively, are independently conserved quantities. Therefore the whole Hilbert space is decomposable into invariant subspaces, i.e., H and U are block diagonal.

The final state of the procedure is an entangled state of the atom-photon system that has components with various numbers of photons, where the maximum possible total number is $N + 1$ (if all atoms have emitted their photons). The probability to find k "right" and l "wrong" photons in the final state, denoted by $p(k,l)$, was calculated for all possible values of k and l and for different values of γt, and from it the overall average "fidelity,"

$$f_{\mathrm{clones}} = \sum_{k+l\geq 2} p'(k,l)\left(\frac{k}{k+l}\right), \qquad (3)$$

was determined. This is the average of the relative frequency of photons with the correct polarization in the final state. Note that in (3) the average is performed only over those cases where there are at least two photons in the final state, i.e., where at least one clone has been produced. $p'(k,l) = p(k,l)/[1 - p(1,0) - p(0,1)]$ is used in order to have proper normalization. Note that $p(0,0)$ is always zero.

That average fidelity for our cloning procedure was compared to the average fidelity that would be achieved by an ensemble of optimal cloners producing the same distribution of numbers of photons, i.e., to

$$f_{\mathrm{opt}} = \sum_{n=2}^{N+1} p'(n)\left(\frac{2n+1}{3n}\right), \qquad (4)$$

where $p'(n) = \sum_{k+l=n} p'(k,l)$. We also made a comparison to the case, where, in addition to the incoming photon, photons are just created randomly, i.e., to the fidelity

$$f_{\mathrm{rand}} = \sum_{n=2}^{N+1} p'(n)\left(\frac{n+1}{2n}\right). \qquad (5)$$

Figure 2(a) shows clearly that the fidelity of our cloning procedure approaches the optimum fidelity for early times. One can also see that for longer interaction times f_{clones} departs from f_{opt} and even becomes lower than f_{rand}. This behavior, which may seem surprising, is due to the fact that for longer times absorption of photons by atoms that have already emitted once and gone to the ground state becomes important. Note that absorption of right photons is favored if there are more such photons present. In particular, also the incoming right photon can be absorbed by an atom that has emitted a wrong photon before, resulting in departure

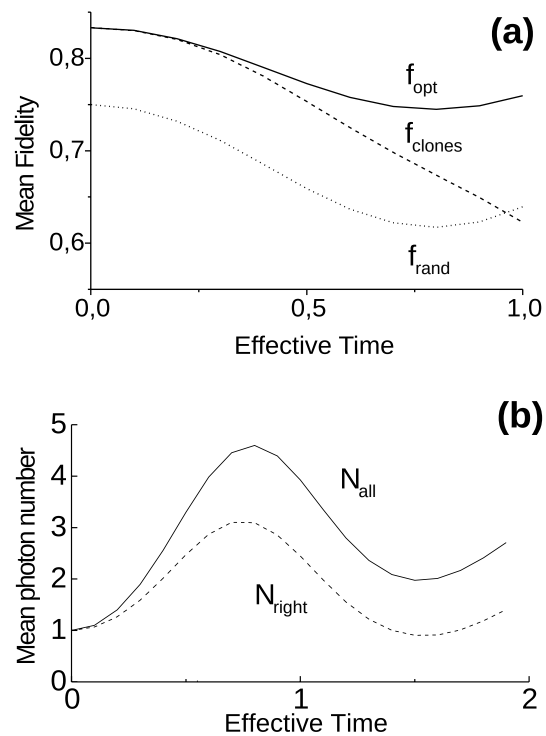

FIG. 2. (a) Dependence on time, measured in units of γt, of f_{opt}, f_{clones}, and f_{rand}, which are the optimum possible fidelity, the fidelity achieved by our three-level cloning procedure, and the fidelity achieved by random photon production, respectively, as defined in Eqs. (3)–(5), for the case of $N = 6$ atoms. It is evident that optimal cloning is achieved in the short-time limit. The behavior for lower atom numbers is the same. (b) Time dependence of the mean number of all photons N_{all} and of the mean number of right photons (i.e., of the same polarization as the incoming photon) N_{right} for the case $N = 6$.

from optimality for later times. The superiority of $f_{\rm rand}$ in that regime is understandable because in our idealized random cloner the incoming photon is always left intact.

Our computations show that the system goes through many emission-reabsorption cycles, though without exhibiting a simple periodicity. As a consequence, over long times $f_{\rm clones}$ oscillates taking values above and below $f_{\rm rand}$, sometimes approaching $f_{\rm opt}$ again.

Figure 2(b), which also illustrates the above-mentioned cyclic behavior of our system, shows the time dependence of the mean number of photons and of the mean number of photons of the correct polarization. For short times, which is the interesting regime from the point of view of cloning, the probability for every individual atom to have already emitted its photon is low. Therefore, in order to produce a reasonable average number of clones in this regime, a large number of atoms is necessary.

The practical realization of this scheme probably requires a cavity in order to achieve the interaction of a single spatial mode of the radiation field with several (or even many) atoms. Trapping several atoms in a cavity could be possible. The atoms could also fly through the cavity [8].

The second scheme for quantum cloning that we want to present is based on stimulated parametric down-conversion (PDC). We will show that optimal cloning can be realized. In PDC a strong light beam is sent through a crystal. There is a certain (very low) probability for a photon from the beam to decay into two photons such that energy and in-crystal momentum are conserved. In type-II PDC the two photons that are created have different polarization. They are denoted as signal and idler.

Figure 3 shows the setup that we have in mind. We consider pulsed type-II frequency-degenerate PDC. It is possible to choose two conjugate directions for the signal and idler beams such that photon pairs that are created

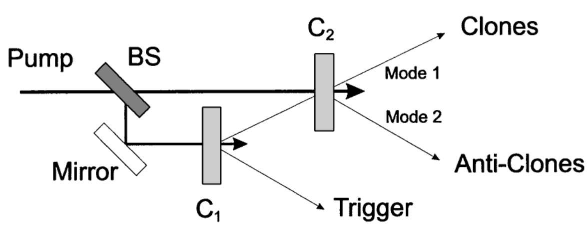

FIG. 3. Setup for optimal cloning by parametric down-conversion [10–12]. The pump pulse is split at the beam splitter (BS). One part of the pump pulse hits the first crystal C_1, where photon pairs are created with a certain rate. One photon from each pair can be used as a trigger. The other photon is the system to be cloned. This photon is directed towards the second crystal C_2, where it stimulates emission of photons of the same polarization along the same direction. The path lengths have to be adjusted in such a way that the DC photon and the second part of the pump pulse reach C_2 simultaneously. The photons in mode 1 are optimal clones of the incoming photon, and the photons in mode 2 are the output of an optimal universal NOT gate. It is interesting to note that, in this scheme, one is actually cloning a photon that is part of an entangled pair.

along these two directions are entangled in polarization [9]. We consider the quasicollinear case (i.e., the two directions almost coincide), so that the transverse motion of the photons in the crystal is not important.

For stimulated emission to work optimally, there has to be maximum overlap of the amplitudes of the incoming photon and of all the photons that are produced in the second crystal. This can be achieved by using a pulsed scheme together with filtering of the photons before detection [13]. The pump pulse can be seen as an active volume that moves through the crystal. If the photons are filtered so much that the smallest possible size of the wave packets detected is substantially bigger than the pump pulse, then there is maximum overlap between different pairs created in the same pulse. Of course, filtering limits the achievable count rates. Moreover the group velocities of pump pulse, signal (V), and idler (H) photons are not all identical. This leads to separations (of the order of a few hundred fs per millimeter in BBO), which have to be kept small compared to the size of the DC-photon wave packets. There is a trade-off between filtering and crystal length, i.e., one can choose narrower filters in order to be able to use a longer crystal (which leads to longer interaction times).

If the above-mentioned conditions are fulfilled, then a single spatial mode (i.e., one mode for the signal and one for the idler photons) approximation can be used. The PDC process can then be described in the limit of a large classical pump pulse, in the interaction picture, by the Hamiltonian

$$H = \gamma(a_{V1}^\dagger a_{H2}^\dagger - a_{H1}^\dagger a_{V2}^\dagger) + \text{H.c.}, \tag{6}$$

where $a_{V1}^\dagger$ is the creation operator for a photon with polarization V propagating along direction 1, etc. The coupling constant and the intensity of the classical pump pulse are contained in γ.

The Hamiltonian H is invariant under simultaneous general SU(2) transformations of the polarization vectors $(a_V^\dagger, a_H^\dagger)$ for modes 1 and 2, while a phase transformation will only change the phase of γ. This makes our cloner universal, i.e., its performance is polarization independent. Therefore it is sufficient to analyze the "cloning" process in one basis.

The time development operator e^{-iHt} clearly factorizes into a $V1$-$H2$ and an $H1$-$V2$ part. Consider cloning starting from N identical photons in the initial state $|\psi_i\rangle = \frac{(a_{V1}^\dagger)^N}{\sqrt{N!}}|0\rangle$. Making use of the disentangling theorem [14], one finds that (cf. [11])

$$\begin{aligned} |\psi_f\rangle &= e^{-iHt}|\psi_i\rangle \\ &= K \sum_{k=0}^{\infty} (-i\Gamma)^k \sqrt{\binom{k+N}{N}} |k+N\rangle_{V1}|k\rangle_{H2} \\ &\quad \times \sum_{l=0}^{\infty} (i\Gamma)^l |l\rangle_{H1}|l\rangle_{V2}, \end{aligned} \tag{7}$$

where $\Gamma = \tanh\gamma t$ and K is a normalizing factor.

The component of this state, which has a fixed number M of photons in mode 1, is proportional to

$$\sum_{l=0}^{M-N} (-1)^l \sqrt{\binom{M-l}{N}} |M-l\rangle_{V1}|l\rangle_{H1}|l\rangle_{V2} \times |M-N-l\rangle_{H2}. \quad (8)$$

This is identical to the state produced by the unitary transformation written down in [15], which can be seen as a special version of the Gisin-Massar cloners [6], that implements optimal universal cloning and the optimal universal NOT gate at the same time. The M photons in mode 1 are the clones, while the $M - N$ photons in mode 2, which act as ancillas for the cloning, are the output of the universal NOT gate, the "anticlones."

In order to see that state (8) is indeed the output of an optimal cloner, let us calculate the relative frequency of photons of the right polarization in mode 1. It is given by

$$f_{\text{clones}}^N(M) = \frac{\sum_{l=0}^{M-N} \binom{M-1}{N}(M-l)}{M\sum_{l=0}^{M-N} \binom{M-l}{N}}. \quad (9)$$

By using $\sum_{k=N}^{M} \binom{k}{N} = \binom{M+1}{N+1}$, it follows that

$$f_{\text{clones}}^N(M) = \frac{NM + N + M}{M(N + 2)}, \quad (10)$$

which is exactly the optimum fidelity for an N to M quantum cloner [5]. A similar calculation demonstrates that the universal NOT is realized in mode 2.

This means that the setup of Fig. 3 works as an ensemble of optimal universal cloning (and universal NOT) machines, producing different numbers of clones and anticlones with certain probabilities. Note that each of the modes can be used as a trigger for the other one, and therefore cloning or anticloning with a fixed number of output systems can be realized by postselection.

We have shown a method of realizing optimal quantum cloning machines. We emphasize that this scheme should be experimentally feasible with current technology. In our group, pair production probabilities of the order of 4×10^{-3} have been achieved with a 76 MHz pulsed laser system (UV power about 0,3 W) and a 1 mm BBO crystal, for 5 nm filter bandwidth. Past experiments show that good overlap of photons originating from different pairs is achieved under these conditions. With detection efficiencies about 10%, this leads to a rate of two-pair detections of the order of one per a few seconds.

A new 300 kHz laser system is currently being set up in our lab. An improvement of the order of $\frac{76}{0,3}$ in the average rate of pairs per pulse is to be expected, for identical pump power. This will also make several-pair events far more likely. This means that production of a few clones with a reasonable rate should be possible.

Here we have presented possible ways of realizing quantum cloning via stimulated emission. We have first discussed a procedure based on three-level systems that could allow the production of large numbers of clones, and could be easier to realize than comparable schemes using quantum gates. We have then shown a scheme for realizing optimal universal cloning based on parametric down-conversion. This scheme should be realizable with current technology.

We would like to thank S. Stenholm for a challenging question that triggered or reanimated our interest in the relationship between quantum cloning and stimulated emission, and V. Bužek for helpful comments; in particular, for bringing the connection between cloning and the universal NOT to our attention. We also thank Č. Brukner, J. I. Cirac, T. Jennewein, J. W. Pan, and H. Weinfurter for helpful comments. This work has been supported by the Austrian Science Foundation (FWF, Projects No. S6504 and No. F1506).

[1] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982).
[2] N. Gisin, Phys. Lett. A **242**, 1 (1998).
[3] N. Herbert, Found. Phys. **12**, 1171 (1982).
[4] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
[5] D. Bruss, A. Ekert, and C. Macchiavello, Phys. Rev. Lett. **81**, 2598 (1998); R. F. Werner, Phys. Rev. A **58**, 1827 (1998); M. Keyl and R. F. Werner, quant-ph/9807010, 1998.
[6] N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997); V. Bužek and M. Hillery, quant-ph/9801009, 1998.
[7] P. W. Milonni and M. L. Hardies, Phys. Lett. **92A**, 321 (1982); L. Mandel, Nature (London) **304**, 188 (1983).
[8] See, e.g., E. Hagley, X. Maitre, G. Nogues, C. Wunderlich, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **79**, 1 (1997); J. Ye, D. W. Vernooy, and H. J. Kimble, quant-ph/9908007.
[9] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. Shih, Phys. Rev. Lett. **75**, 4337 (1995).
[10] Stimulated emission in continuous-wave PDC by classical incoming light was studied by Z. Y. Ou, L. J. Wang, X. Y. Zou, and L. Mandel [Phys. Rev. A **41**, 1597 (1990)].
[11] Stimulated emission in pulsed PDC was also considered by F. De Martini [Phys. Rev. Lett. **81**, 2842 (1998)] for a different purpose, namely, the generation of Schrödinger cat states. In the meantime, first experimental results have been presented [V. Mussi and F. De Martini, in Proceedings of III Adriatico Research Conference on Quantum Interferometry, Trieste, 1999 (to be published)].
[12] A similar experimental setup can be used for an absolute measurement of the spectral radiance of a photon beam [see A. Migdall, Phys. Today **52**, No. 1, 41 (1999)].
[13] A. Zeilinger, M. A. Horne, H. Weinfurter, and M. Zukowski, Phys. Rev. Lett. **78**, 3031 (1997).
[14] M. J. Collett, Phys. Rev. A **38**, 2233 (1988); D. F. Walls and G. I. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1995), Chap. 5.
[15] V. Bužek, M. Hillery, and R. F. Werner, quant-ph/9901053; see also N. Gisin and S. Popescu, quant-ph/9901072.