Relaxation and Diffusion for the Kicked Rotor

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The dynamics of the kicked rotor, which is a paradigm for a mixed system, where the motion in some parts of phase space is chaotic and in other parts is regular, is studied statistically. The evolution operator of phase space densities in the chaotic component is calculated in the presence of noise, and the limit of vanishing noise is taken in the end. The relaxation rates to the equilibrium density are calculated analytically within an approximation that improves with increasing stochasticity. The results are tested numerically. A global picture is presented of relaxation to the equilibrium density in the chaotic component when the system is bounded and to diffusive behavior when it is unbounded.

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Statistical analysis is most appropriate for the exploration of global properties of systems that exhibit complicated dynamics [1–4]. For chaotic systems, $\hat{U}$, the evolution operator of distributions of phase space trajectories, which is sometimes called the Frobenius-Perron (FP) operator, describes the statistical properties of the dynamics. For many idealized systems exponential relaxation to the equilibrium density takes place. This was established rigorously for hyperbolic systems (*A* systems), such as the baker map [2–6]. The relaxation rates relate to the Ruelle resonances that are poles of the matrix elements of the resolvent $\hat{R} = (z - \hat{U})^{-1}$ in a space of functions that are sufficiently smooth [5]. These poles are inside the unit circle in the complex z plane while the spectrum of $\hat{U}$ is confined to the unit circle because of unitarity. Most physically realistic models are not hyperbolic, but mixed where the phase space consists of chaotic and regular components. For mixed systems sticking to regular regions takes place. If the regular regions are small this effect is negligible for finite time, which may be long, much longer than the time relevant to the experiment. In this Letter the FP operator and the relevant approximate relaxation rates are calculated analytically and numerically for the chaotic component of the kicked rotor that is a mixed system [7].

The kicked rotor is a paradigm [8] for chaotic behavior of systems where one variable may be either bounded or unbounded in phase space. If it is unbounded, diffusion is found for the classical system [8,9]. In quantum mechanics this diffusion is suppressed by a mechanism similar to Anderson localization [10]. The kicked rotor is defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2} J^2 + K \cos\theta \sum_n \delta(t - n), \qquad (1)$$

where J is the angular momentum, θ is the conjugate angle ($0 \le \theta < 2\pi$), and K is the stochasticity parameter. Its equations of motion reduce to the standard map $\bar{\theta} = \theta + \bar{J}$ and $\bar{J} = J - K\sin\theta$, where (θ, J) and $(\bar{\theta}, \bar{j})$ are the angle and the angular momentum before a kick, and just before the next kick, respectively. For $K > K_c \approx 0.9716$, diffusion in phase space was found.

In this paper the FP operator will be calculated for the kicked rotor on the torus: $(0 \le J < 2\pi s; 0 \le \theta < 2\pi)$, where s is an integer. The operator is defined in the space spanned by the Fourier basis:

$$\phi_{km} = (J\theta \,|\, km) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi s}} \exp(im\theta) \exp\left(i \frac{kJ}{s}\right). \qquad (2)$$

The FP operator was studied rigorously for the hyperbolic systems and many of its properties are known [2–6]. It is a unitary operator in $\mathcal{L}^2$, the Hilbert space of square integrable functions. Therefore its resolvent

$$\hat{R}(z) = \frac{1}{z - \hat{U}} = \frac{1}{z} \sum_{n=0}^{\infty} \hat{U}^n z^{-n} \qquad (3)$$

is singular on the unit circle in the complex z plane. The matrix elements of $\hat{R}$ are discontinuous there, and one finds a jump between two Riemann sheets. The sum (3) is convergent for $|z| > 1$, therefore it identifies the physical sheet as the one connected with this region. The Ruelle resonances are the poles of the matrix elements of the resolvent, on the Riemann sheet, extrapolated from $|z| > 1$ [6]. These describe the decay of *smooth* probability distribution functions to the invariant density in a coarse grained form [3]. In spite of the solid mathematical theory, there are very few examples where the Ruelle resonances were calculated for specific systems [3,6]. For the baker map it is easy to see that, as the resonances approach the unit circle, corresponding to slower decay, they are associated with coarser resolution in phase space [6]. For the kicked rotor (1) the operation of $\hat{U}$ on a phase space density ρ is $\hat{U}\rho(\theta, J) = \rho(\theta - J, J + K\sin(\theta - J))$. To make the calculation, well-defined noise is added to the system. If noise that conserves J and leads to diffusion in θ is added to the free motion, the matrix elements of $\hat{U}$ in the basis (2) are

$$(k_2 m_2|\hat{U}|k_1 m_1) = J_{m_2 - m_1}\left(\frac{k_1 K}{s}\right) \times \exp\left(-\frac{\sigma^2}{2} m_2^2\right) \delta_{k_2 - k_1, m_2 s}. \qquad (4)$$

For $\sigma = 0$ the operator is unitary as required. It is shown explicitly that addition of the noise acts effectively as coarse graining and the resulting evolution operator is not unitary (see also [11]). For a large stochasticity parameter K, it is *shown* here, by a direct order by order calculation without any additional assumptions (of the nature made in [8,9]), that in the Fourier basis (2) the *slowest* relaxation modes, in the limit of infinitesimal noise, are found to be identical to the modes of the diffusion operator [7]. Also the *fast* relaxation modes are calculated analytically in this paper, and the approximate analytical results are tested numerically [7]. These modes are not related to the spectrum of the FP operator that is confined to the unit circle. We believe that we found all relaxation rates for distribution functions that can be expanded in terms of the basis functions (2). The immediate question is how is it possible that this description, which was established only for hyperbolic systems, holds for a mixed system. It is clearly approximate, and holds for large values of the stochasticity parameter K, since most of the phase space is then covered by the chaotic component. The physical reason for the decay of correlations is that, in a chaotic system, the stretching and folding mechanisms lead to a persistent flow in the direction of functions with finer details, namely, larger $|k|$ and $|m|$ in our case. Consequently, the projection on a given function, for example, one of the basis functions (2) in our case, decays [12]. The crucial point is that this function should be sufficiently smooth. This argument should hold as an approximation also for the chaotic component of mixed systems. For smaller values of K the weight of the regular regions increases. In such a situation, in the limit of increasing resolution, the resonances related to the regular component are expected to move to the unit circle in the complex z plane, corresponding to the quasiperiodic motion, while the resonances associated with the chaotic component stay inside the unit circle [13].

How is the FP operator related to the quantum mechanical evolution operator? It was shown numerically for the baker map that if both operators are calculated with finite resolution they exhibit the same Ruelle resonances [11]. Noise and coarse graining are introduced in field theoretical treatment of chaotic systems [14,15]. Since the FP operator plays an important role in these theories, our work is of relevance there. It also justifies some assumptions made in the calculation of the typical localization length for the kicked rotor [16,17].

We turn now to calculate the Ruelle resonances for the kicked rotor with the help of the evolution operator (4). The calculation will be done for finite noise σ and then the limit $\sigma \to 0$ will be taken. These are the poles of matrix elements $R_{12} = (k_1 m_1|\hat{R}(z)|k_2 m_2)$ of the resolvent operator $\hat{R}$ of (3) when analytically continued from outside to inside the unit circle in the complex plane. It is useful to introduce $\hat{R}'(z) = 1/(1 - z\hat{U}) = \sum_{n=0}^{\infty} z^n \hat{U}^n$ that is convergent inside the unit circle, because $\|z\hat{U}\| \leq 1$. Its matrix elements are $R'_{12} = [k_1 m_1|\hat{R}'(z)|k_2 m_2] = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = (k_1 m_1|\hat{U}^n|k_2 m_2)$. The relation between the matrix elements inside and outside of the unit circle implies that if z_c is a singularity of R_{12} then $1/z_c$ is a singular point of R'_{12}. Consequently, the first singularity of the analytic continuation of $R'_{12}(z)$ from inside to outside the unit circle gives the first singularity one encounters when analytically continuing $R_{12}(z)$ from outside to inside the unit circle, i.e., it is just the leading nontrivial resonance. It is determined from the fact that it is the radius of convergence r of the series for R'_{12} given by the Cauchy-Hadamard theorem: $r^{-1} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$ [18].

The calculation of the coefficients $a_n = (k0|\hat{U}^n|k0)$ is performed using the resolution of the identity [introducing intermediate $|k_i m_i)(k_i m_i|$], and then substitution of (4) and summation over the k_i, leading to

$$a_n = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_{n-1}} \prod_{l=1}^{n} J_{M_l^-}\left(\frac{kK}{s} - KM_{l-1}^+\right) \times e^{-(\sigma^2/2)m_l^2} \delta_{M_{n-1}^+,0}, \tag{5}$$

where $m_0 = m_n = 0$, while $M_l^+ = \sum_{i=0}^{l} m_i$ and $M_l^- = m_{l-1} - m_l$. The calculation is performed for large s and K and the limits are taken in order [7]:

$$\begin{aligned} &(1) \quad s \to \infty, \\ &(2) \quad K \to \infty, \\ &(3) \quad \sigma \to 0. \end{aligned} \tag{6}$$

For a sufficiently low mode so that $0 < kK/s \ll 1$, the leading order term in K/s and $1/\sqrt{K}$ is

$$a_n \sim \left[1 - \frac{k^2 K^2}{4s^2}[1 - 2J_2(K)e^{-\sigma^2}]\right]^n. \tag{7}$$

The resonance closest to the unit circle, z_k is the inverse of the radius of convergence. Here $z_k = e^{-(k^2/s^2)D(K)}$, with

$$D(K) = \frac{K^2}{4}[1 - 2J_2(K)e^{-\sigma^2}], \tag{8}$$

which is just the value of the diffusion coefficient $D(K)$ found in [19]. In the limit of $\sigma \to 0$ these are the relaxation rates in the diffusion equation. The analysis of the off-diagonal matrix elements $a_n = (km|\hat{U}^n|k'm')$ leads to the same result.

In order to obtain the fast relaxation rates we have to calculate matrix elements which do not exhibit slow relaxation because such relaxation, if present, dominates the long time behavior. For this purpose we calculated the relaxation rates of disturbances from invariant density which involve functions from the subspace $\{|0, m)\}$ with $m \neq 0$ and calculate $a_n = (0m|\hat{U}^n|km')$. Again, the resolution of the identity is introduced [introducing intermediate $|k_i m_i)(k_i m_i|$], and summation over the k_i yields a nonvanishing result only if $k/s \equiv q$ is an integer. The expression for a_n is found to be independent of s. The resulting resonances (for large K) are $\tilde{z}_p = \sqrt{J_{2p}(pK)\exp(-\sigma^2 p^2/2)}$, where $p = |m|$ or $p = |\tilde{q}|$, with $\tilde{q} = q$ if $q \neq 0$ and $\tilde{q} = m'$ if $q = 0$, depending upon which choice gives the larger absolute value [7]. For $q = m' = 0$ and $m \neq 0$, one finds $a_n = 0$. If $m = m' = q = 0$ the only contribution is when all m_i vanish and then $a_n = 1$ for all n, resulting in the resonance $z = 1$, corresponding to equilibrium.

The FP operator is the evolution operator $\hat{U}$ in the limit of vanishing noise. Therefore the Ruelle resonances are the poles of matrix elements of the resolvent $\hat{R}$ in this limit. They form several groups. There is $z_0 = 1$, which is related to the equilibrium state. The resonances corresponding to the relaxation modes related to the diffusion in the angular momentum are

$$z_k = \exp\left(-\frac{k^2K^2}{4s^2}[1 - 2J_2(K)]\right). \tag{9}$$

The resonances related to fast relaxation in the θ direction are

$$\tilde{z}_p = \sqrt{J_{2p}(pK)}. \tag{10}$$

In certain cases this result does not hold for small intervals around the values of K, where $J_{2p}(pK) = 0$ [7]. The relaxation rates are $\gamma_k = |\ln z_k|$ and $\tilde{\gamma}_p = |\ln|\tilde{z}_p||$.

The analytical results that were obtained as the leading terms in an expansion in powers of $1/\sqrt{K}$ were tested numerically for finite K and $\sigma = 0$. For this purpose the correlation function $C_{fg}(n) = (f|\hat{U}^n|g)$ was calculated numerically. For distributions g and f from the Fourier basis (2), projected on the chaotic component, the relaxation rates are expected to take the values γ_k or $\tilde{\gamma}_p$. For the diffusive modes, one expects $\gamma_k = (k^2/s^2)D(K)$, where $D(K)$ is the diffusion coefficient (8) with $\sigma = 0$. The values of $D(K)$ were extracted from this relation for various values of k and s and are presented in Fig. 1. For large values of K, excellent agreement with the theory is found: the value of D is found to be independent of k and s and it agrees with (8). For relatively smaller values of K, the value of the diffusion coefficient for some values of K is larger than the one that is theoretically predicted. The theoretical errors were estimated from the next term of the formula of Rechester and White for the diffusion coefficient [19]. In order to observe the rapidly relaxing modes the correlation function C_{fg} was calculated for $g = \phi_{km'}$ so that $q = k/s$ is an integer and $f = \phi_{0m}$. In Fig. 2 the numerical estimate for $\tilde{\gamma}_p$ is compared with the theoretical prediction obtained from (10). The error in the theoretical prediction is estimated as the value of the next order contribution to a_n. The main reason for disagreement between the theory and the numerical simulations is the sticking to the islands of stability and accelerator modes [20].

Finite noise leads to the effective truncation of the evolution operator (4) . In the basis (2) it means that it results in limited resolution. Moreover, for $\sigma > 0$ the operator $\hat{U}$ is nonunitary. The approximate eigenvalues of $\hat{U}$ given by (4) that were found in this work are 1, z_k, and $\tilde{z}_p$. Because

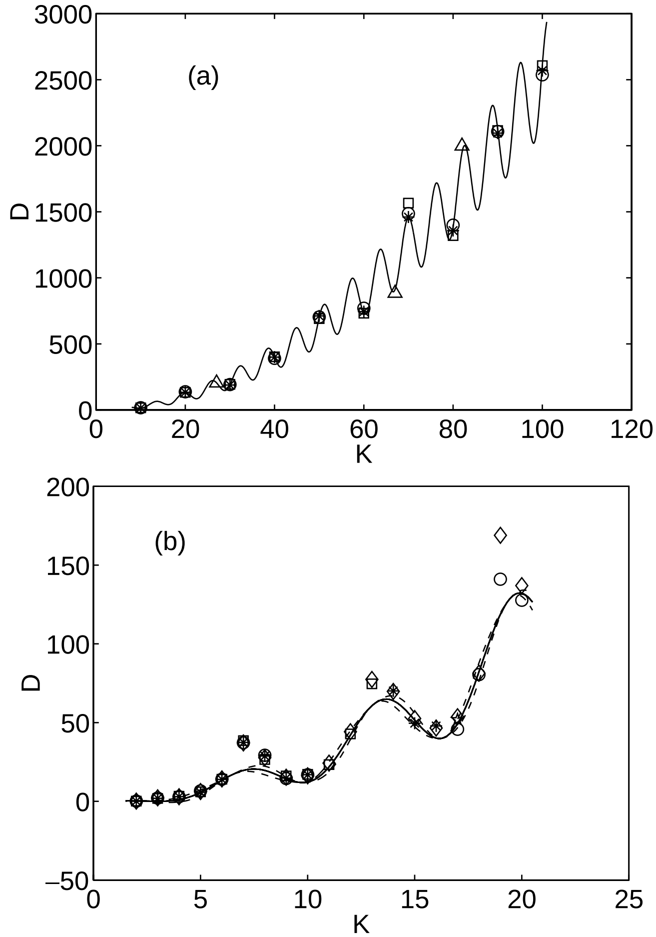

FIG. 1. The diffusion coefficient D for (a) $10 \leq K \leq 100$ and (b) $2 \leq K \leq 20$, as extracted from the relaxation rates: the first mode (squares), the second mode (stars), the fifth mode (circles), and off-diagonal correlation functions (triangles) compared to the theoretical value (solid line). The dashed line represents the approximate error. The values of D obtained by direct simulation of propagation of trajectories are marked by diamonds.

of the effective truncation, ψ_γ (the eigenfunction of $\hat{U}$) can be expanded in terms of the basis states (2). The relaxation rates of these eigenstates are $-\ln(z_k)$ and $-\ln(|\tilde{z}_p|)$. In the limit $\sigma \rightarrow 0$ the evolution operator is unitary, and the functions ψ_γ approach some generalized functions while z_k and $\tilde{z}_p$ approach the values (9) and (10). These are the Ruelle resonances similar to the ones found for hyperbolic systems such as the baker map [6]. Here, noise was used in order to make the analytical calculations possible, and to develop a physically transparent description of the evolution. In real experiments some level of noise is present, therefore the results in the presence of noise are of experimental relevance.

In summary, the Ruelle resonances that were found rigorously for hyperbolic systems can be used for an approximate description of relaxation and transport in the chaotic component of mixed systems. The relaxation of distributions in phase space to the invariant density takes place

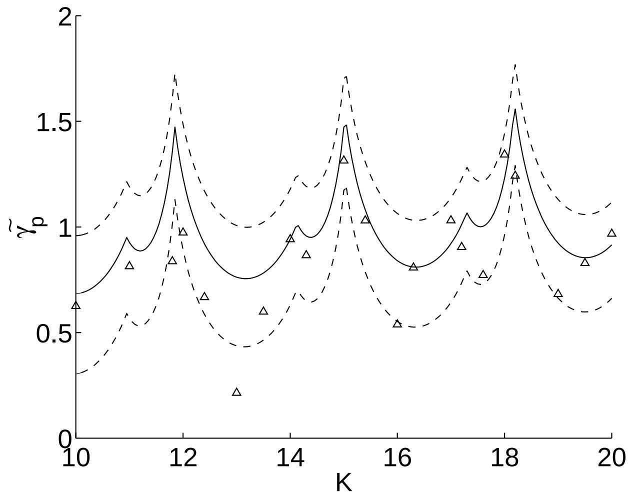

FIG. 2. The fast relaxation rates $\tilde{\gamma}_p$ for $f = \phi_{01}, g = \phi_{02}$ (triangles) compared to the theoretical value (solid line). The dashed lines denote the theoretically estimated error. Here we used $s = 1$ and $N = 10^8$.

in stages. The inhomogeneity in θ decays with the rapid relaxation rates $\tilde{\gamma}_p$ and then relaxation of the inhomogeneities in the J direction takes place with the relaxation rates of the diffusion equation. In the limit $s \rightarrow \infty$ the inhomogeneity in θ relaxes and then diffusion in the momentum direction takes place.

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