Interpretation of the Nonextensivity Parameter q in Some Applications of Tsallis Statistics and Lévy Distributions

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The nonextensivity parameter q occurring in some of the applications of Tsallis statistics (known also as index of the corresponding Lévy distribution) is shown to be given, in the $q > 1$ case, entirely by the fluctuations of the parameters of the usual exponential distribution.

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There is an enormous variety of physical phenomena described most economically (by introducing only one new parameter q) and adequately by the so called nonextensive statistics introduced some time ago by Tsallis [1]. They include all situations characterized by long-range interactions, long-range microscopic memories, and space-time (and phase space as well) (multi)fractal structure of the process (cf. [1] for details). The high energy physics applications of nonextensive statistics are quite recent but are already numerous and still growing; cf. Refs. [2–8]. All examples mentioned above have one thing in common: the central formula employed is the following powerlike distribution:

$$G_q(x) = C_q\left[1 - (1 - q)\frac{x}{\lambda}\right]^{\frac{1}{1-q}}, \qquad (1)$$

which is just a one parameter generalization of the Boltzmann-Gibbs exponential formula to which it converges for $q \to 1$:

$$G_{q=1} = g(x) = c\exp\left[-\frac{x}{\lambda}\right]. \qquad (2)$$

When $G_q(x)$ is used as probability distribution (Lévy distribution) of the variable $x \in (0,\infty)$ (which will be the case we are interested in here), the parameter q is limited to $1 \le q < 2$. For $q < 1$, the distribution $G_q(x)$ is defined only for $x \in [0, \lambda/(1 - q)]$. For $q > 1$ the upper limit comes from the normalization condition (to unity) for $G_q(x)$ and from the requirement of the positivity of the resulting normalization constant C_q. However, if one demands in addition that the mean value of $G_q(x)$ is well defined, i.e., that $\langle x\rangle = \lambda/(3 - 2q) < \infty$ for $x \in (0,\infty)$, then q is further limited to $1 \le q < 1.5$ only. In spite of numerous applications of the Lévy distribution $G_q(x)$, the interpretation of the parameter q is still an open issue. In this Letter we demonstrate, on the basis of our previous application of the Lévy distribution to cosmic rays [2], that this Lévy distribution $G_q(x)$ (1) emerges in a natural way from the fluctuations of the parameter $1/\lambda$ of the original exponential distribution (2) and that the parameters of its distribution $f(1/\lambda)$ define parameter q in a unique way.

Let us first briefly summarize the result of [2]. Analyzing experimental distributions $dN(x)/dx$ of depths x of interactions of hadrons from cosmic ray cascades in the emulsion chambers, we have shown that the so called *long flying component* [manifesting itself in apparently unexpected nonexponential behavior of $dN(x)/dx$] is just a manifestation of the Lévy distribution $G_q(x)$ with $q = 1.3$. This result must be confronted with our earlier analysis of the same phenomenon [9]. We have demonstrated there that distributions $dN(x)/dx$ can be also described by the fluctuation of the corresponding cross section $\sigma = Am_N\frac{1}{\lambda}$ (where A denotes mass number of the target, m_N is the mass of the nucleon, and λ is the corresponding mean free path). The fluctuation of this cross section (i.e., in effect, fluctuations of the quantity $1/\lambda$) with relative variance

$$\omega = \frac{\langle\sigma^2\rangle - \langle\sigma\rangle^2}{\langle\sigma\rangle^2} \ge 0.2 \qquad (3)$$

allow us to describe the nonexponentiality of the experimental data as well as the distribution $G_{q=1.3}(x)$ mentioned above. We therefore argue that these two numerical examples show that fluctuations of the parameter $1/\lambda$ in the $g(x;\lambda)$ result in the Lévy distributions $G_q(x;\lambda)$.

Actually the above quoted example from cosmic ray physics is not the only one known at present in the field of high energy collisions. It turns out [3–5] that distributions of transverse momenta $dN(p_T)/dp_T$ are best described by a slightly nonexponential distribution $G_q(p_T)$ of the Lévy type with $q = 1.01$–1.2 depending on the situation considered. The usual exponential distribution $dN(p_T)/dp_T = g(p_T) \sim \exp(-\sqrt{m^2 + p_T^2}/kT)$ contains as a main parameter the inverse temperature $\beta = 1/kT$ and the above mentioned numerical results leading to $G_{q=1.01-1.2}(p_T)$ can again be understood as a result of a fluctuation of inverse temperature β in the usual exponential formula $g(p_T)$. This point is of special interest because of recent discussions on the dynamical possibility of temperature fluctuations in some collisions; cf. Refs. [10–12]. Later on we shall use it to illustrate our results concerning q.

To recapitulate, we claim that (for $q > 1$) the parameter q is nothing but a measure of fluctuations present in Lévy distributions $G_q(x)$ describing particular processes

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under consideration. To make our statement more quantitative, let us analyze the influence of fluctuations of parameter $1/\lambda$ which are present in the exponential formula $g(x) \sim \exp(-x/\lambda)$ on the final result. Our aim will be a deduction of the form of the function $f(1/\lambda)$ which leads from an exponential distribution $g(x)$ to powerlike Lévy distribution $G_q(x)$ and which describes fluctuation about the mean value $1/\lambda_0$, i.e., such that

$$G_q(x;\lambda_0) = C_q\left(1 + \frac{x}{\lambda_0}\frac{1}{\alpha}\right)^{-a} = C_q\int_0^\infty \exp\left(-\frac{x}{\lambda}\right)f\left(\frac{1}{\lambda}\right)d\left(\frac{1}{\lambda}\right), \quad (4)$$

where for simplicity we have introduced the abbreviation $\alpha = \frac{1}{q-1}$. From the representation of the Euler gamma function we have [13]

$$\left(1 + \frac{x}{\lambda_0}\frac{1}{\alpha}\right)^{-a} = \frac{1}{\Gamma(\alpha)}\int_0^\infty d\xi\, \xi^{\alpha-1} \times \exp\left[-\xi\left(1 + \frac{x}{\lambda_0}\frac{1}{\alpha}\right)\right]. \quad (5)$$

Changing variables under the integral in such a way that $\frac{\xi}{\lambda_0}\frac{1}{\alpha} = \frac{1}{\lambda}$ one immediately obtains Eq. (4) with $f(1/\lambda)$ given by the following gamma distribution:

$$f\left(\frac{1}{\lambda}\right) = f_\alpha\left(\frac{1}{\lambda},\frac{1}{\lambda_0}\right) = \frac{1}{\Gamma(\alpha)}(\alpha\lambda_0)\left(\frac{\alpha\lambda_0}{\lambda}\right)^{\alpha-1}\exp\left(-\frac{\alpha\lambda_0}{\lambda}\right) \quad (6)$$

with mean value

$$\left\langle\frac{1}{\lambda}\right\rangle = \frac{1}{\lambda_0} \quad (7)$$

and variation

$$\left\langle\left(\frac{1}{\lambda}\right)^2\right\rangle - \left\langle\frac{1}{\lambda}\right\rangle^2 = \frac{1}{\alpha\lambda_0^2}. \quad (8)$$

Notice that with increasing α variance (8) decreases and asymptotically (for $\alpha \to \infty$, i.e., for $q \to 1$) the gamma distribution (6) becomes a delta function $\delta(\lambda - \lambda_0)$. The relative variance [cf. Eq. (3)] for this distribution is given by

$$\omega = \frac{\langle(\frac{1}{\lambda})^2\rangle - \langle\frac{1}{\lambda}\rangle^2}{\langle\frac{1}{\lambda}\rangle^2} = \frac{1}{\alpha} = q - 1. \quad (9)$$

We see therefore that, indeed, the parameter q in the Lévy distribution $G_q(x)$ describes the relative variance of the parameter $1/\lambda$ present in the exponential distribution $g(x;\lambda)$.

Some remarks on the numerical results quoted before [2,9] are in order here. Notice that the value of $q = 1.3$ for cosmic ray distribution $dN(x)/dx$ obtained in [2] leads to the relative variance of the cross section $\omega = 0.3$, whereas in [9] we have reported value $\omega' = 0.2$. This discrepancy has its origin in the fact that in numerical calculations in [9] we have used a symmetric Gaussian distribution to describe fluctuations of the cross section, whereas the relation (9) has been obtained for fluctuations described by gamma distribution. In the Gaussian approximation we expect that

$$\frac{q-1}{q^2} < \omega' < q - 1, \quad (10)$$

where lower and upper limits are obtained by normalizing the variance of the $f(1/\lambda)$ distribution to the modial [equal to $(2 - q)/\lambda_0$] and mean (equal to $1/\lambda_0$) values, respectively. Therefore for $q = 1.3$ one should expect that $0.18 < \omega' < 0.3$, which is exactly the case.

Let us now proceed to the above mentioned analysis of transverse momentum distributions in heavy ion collisions, $dN(p_T)/dp_T$ [4,5]. It is interesting to notice that the relatively small value $q \simeq 1.015$ of the nonextensive parameter obtained there, if interpreted in the same spirit as above, indicates that rather large relative fluctuations of temperature, of the order of $\Delta T/T \simeq 0.12$, exist in nuclear collisions. It could mean therefore that we are dealing here with some fluctuations existing in small parts of the system in respect to the whole system (according to interpretation of [12]) rather than with fluctuations of the event-by-event type in which, for large multiplicity N, fluctuations $\Delta T/T = 0.06/\sqrt{N}$ should be negligibly small [10].

We now propose a general explanation of the meaning of the function $f(\chi)$ describing fluctuations of some variable χ. In particular, we are interested in why, and under what circumstances, it is the gamma distribution that describes fluctuations. To this end let us start with the well known equation for the variable χ, which in the Langevin formulation has the form [14]

$$\frac{d\chi}{dt} + \left[\frac{1}{\tau} + \xi(t)\right]\chi = \phi = \text{const} > 0. \quad (11)$$

Let us concentrate for our purposes on the stochastic process which is defined by the *white Gaussian noise* $\xi(t)$, with ensemble mean,

$$\langle\xi(t)\rangle = 0 \quad (12)$$

and correlator $\langle\xi(t)\xi(t+\Delta t)\rangle$, which for sufficiently fast changes is equal to

$$\langle\xi(t)\xi(t+\Delta t)\rangle = 2D\delta(\Delta t). \quad (13)$$

Constants τ and D define, respectively, the mean time for changes and their variance by means of the following conditions:

$$\langle\chi(t)\rangle = \chi_0\exp\left(-\frac{t}{\tau}\right), \qquad \langle\chi^2(t=\infty)\rangle = \frac{1}{2}D\tau. \quad (14)$$

Thermodynamical equilibrium is assumed here (i.e., $t \gg \tau$, in which case the influence of the initial condition χ_0 vanishes and the mean squared of χ has value corresponding to the state of equilibrium). Making use of the Fokker-Planck equation [15]

$$\frac{df(\chi)}{dt} = -\frac{\partial}{\partial\chi}K_1 f(\chi) + \frac{1}{2}\frac{\partial^2}{\partial\chi^2}K_2 f(x) \qquad (15)$$

we get for the distribution function the expression

$$f(\chi) = \frac{c}{K_2(\chi)}\exp\left[2\int_0^{\chi} d\chi' \frac{K_1(\chi')}{K_2(\chi')}\right], \qquad (16)$$

where the constant c is defined by the normalization condition for $f(x)$: $\int_0^\infty d\chi\, f(\chi) = 1$. K_1 and K_2 are the intensity coefficients which for the process defined by Eq. (11) are equal to (cf., for example, [16])

$$K_1(\chi) = \phi - 2\frac{\chi}{\tau} + D\chi\,,$$
$$K_2(\chi) = 2D\chi^2. \qquad (17)$$

It means therefore that as a result we have the following distribution function:

$$f(\chi) = \frac{1}{\Gamma(\alpha)}\mu\left(\frac{\mu}{\chi}\right)^{\alpha-1}\exp\left(-\frac{\mu}{\chi}\right), \qquad (18)$$

which is nothing but a gamma distribution of variable $1/\chi$ depending on two parameters:

$$\mu = \frac{\phi}{D}, \qquad \alpha = \frac{1}{\tau D}. \qquad (19)$$

Returning to the q notation [cf. Eq. (4)] we have therefore

$$q = 1 + \tau D; \qquad (20)$$

i.e., the parameter of nonextensivity is given by the parameter D describing the *white noise* and by the damping constant τ. This means then that the relative variance $\omega(1/\chi)$ of distribution (18) is [as in Eq. (9)] given by τD.

As illustration of the genesis of Eq. (11) used to derive Eq. (20), we turn once more to the fluctuations of temperature [10–12] discussed before (i.e., to the situation when $\chi = T$). Suppose that we have a thermodynamic system, in a small (mentally separated) part of which the temperature fluctuates with $\Delta T \sim T$. Let $\xi(t)$ describe stochastic changes of temperature in time. If the mean temperature of the system $\langle T\rangle = T_0$, then, as a result of fluctuations in some small selected region, the actual temperature T' equals

$$T' = T_0 - b\xi(t)T\,, \qquad (21)$$

where the constant b is defined by the actual definition of the stochastic process under consideration, i.e., by $\xi(t)$, which is assumed to satisfy conditions given by Eqs. (12) and (13). The inevitable exchange of heat between this selected region and the rest of the system leads to the equilibration of the temperature. The corresponding process of heat conductance is described by the equation [17]

$$c_p\rho\frac{\partial T}{\partial t} - a(T' - T) = 0\,, \qquad (22)$$

where c_p, ρ, and a are, respectively, the specific heat, density, and the coefficient of the external conductance. Using T' as defined in (21) we finally get the linear differential equation (11) for the temperature T with coefficients $\tau = \frac{c_p\rho}{a}$, $\phi = \frac{a}{c_p\rho}T_0 = T_o/\tau$, and $b = \tau$:

$$\frac{\partial T}{\partial t} + \left[\frac{a}{c_p\rho} + \frac{a}{c_p\rho}b\xi(t)\right]T = \frac{a}{c_p\rho}T_0\,. \qquad (23)$$

This result demonstrates clearly that one can think of a deep physical interpretation of the parameter q of the corresponding Lévy distribution describing the distributions of the transverse momenta mentioned before. In this respect our work differs from works in which $G_q(x)$ is shown to be connected with $G_{q=1}(x) = g(x)$ by the so called Hilhorst integral formula [the trace of which is our Eq. (5)] [13,18] but without discussing the physical context of the problem. Our original motivation was to understand the apparent success of Tsallis statistics (i.e., the situations in which $q > 1$) in the realm of high energy collisions.

To summarize, if fluctuations of the variable χ can be described in terms of the Langevin formulation, their distribution function $f(1/\chi)$ satisfies the Fokker-Planck equation and is therefore given by the gamma distribution in the variable $1/\chi$. Such fluctuations of the parameter $1/\chi$ in the exponential formula of physical interest, $g(x/\chi)$, lead immediately to a Lévy distribution $G_{q>1}(x/\chi)$ with q parameter given by the relative variance of the fluctuations described by $f(1/\chi)$. It should be stressed that in this way we address the interpretation of only very limited cases of applications of Tsallis statistics. They belong to the category in which the power laws physically appear as a consequence of some continuous spectra within appropriate integrals. It does not touch, however, a really *hard* case of applicability of Tsallis statistics, namely, when *zero* Lyapunov exponents are involved [19]. Nevertheless, this allows us to interpret some nuclear collisions data in terms of fluctuations of the inverse temperature, providing thus an important hint to the origin of some systematics in the data, understanding of which is crucial in the search for the new state of matter: the quark gluon plasma [4,11].

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