Peres-Horodecki Separability Criterion for Continuous Variable Systems

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The Peres-Horodecki criterion of positivity under partial transpose is studied in the context of separability of bipartite continuous variable states. The partial transpose operation admits, in the continuous case, a geometric interpretation as mirror reflection in phase space. This recognition leads to uncertainty principles, stronger than the traditional ones, to be obeyed by all separable states. For all bipartite Gaussian states, the Peres-Horodecki criterion turns out to be a necessary and sufficient condition for separability.

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Entanglement or inseparability is central to all branches of the emerging field of quantum information and quantum computation [1]. A particularly elegant criterion for checking if a given state is separable or not was proposed by Peres [2]. This condition is necessary and sufficient for separability in the 2×2 and 2×3 dimensional cases, but ceases to be a sufficient condition in higher dimensions, as shown by Horodecki [3].

While a major part of the effort in quantum information theory has been in the context of systems with finite number of Hilbert space dimensions, more specifically the qubits, recently there has been much interest in the canonical continuous case [4–9]. We may mention, in particular, the experimental realization of quantum teleportation of coherent states [10]. It is therefore important to be able to know if a given state of a bipartite canonical continuous system is entangled or separable.

With increasing Hilbert space dimension, any test for separability will be expected to become more and more difficult to implement in practice. In this paper we show that in the limit of infinite dimension, corresponding to continuous variable bipartite states, the Peres-Horodecki criterion leads to a test that is extremely easy to implement [11]. Central to our work is the recognition that the partial transpose operation acquires, in the continuous case, a beautiful geometric interpretation as *mirror reflection in the Wigner phase space*. Separability forces on the second moments (uncertainties) a restriction that is stronger than the traditional uncertainty principle; even commuting variables need to obey an uncertainty relation. This restriction is used to prove that the Peres-Horodecki criterion is a necessary and sufficient separability condition for all bipartite Gaussian states.

Consider a single mode described by annihilation operator $\hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}$, obeying the standard commutation relation $[\hat{q}, \hat{p}] = i$, which is equivalent to $[\hat{a}, \hat{a}^\dagger] = 1$. There is a one-to-one correspondence between density operators and c-number Wigner distribution functions $W(q, p)$ [12]. The latter are real functions over the phase space and satisfy an additional property coding the non-negativity of the density operator. It follows from the definition of Wigner distribution that the transpose operation T, which takes every $\hat{\rho}$ to its transpose $\hat{\rho}^T$, is *equivalent to* a mirror reflection in phase space:

$$\hat{\rho} \to \hat{\rho}^T \Longleftrightarrow W(q, p) \to W(q, -p). \quad (1)$$

Mirror reflection is not a canonical transformation in phase space, and cannot be implemented unitarily in the Hilbert space. This is consistent with the fact that while T is linear at the density operator level, it is antilinear at the state vector or wave function level (time reversal).

Now consider a bipartite system of two modes described by annihilation operators $\hat{a}_1 = (\hat{q}_1 + i\hat{p}_1)/\sqrt{2}$ and $\hat{a}_2 = (\hat{q}_2 + i\hat{p}_2)/\sqrt{2}$. Let Alice be in possession of mode 1 and let mode 2 be in the possession of Bob. By definition, a quantum state $\hat{\rho}$ of the bipartite system is separable if and only if $\hat{\rho}$ can be expressed in the form

$$\hat{\rho} = \sum_j p_j \hat{\rho}_{j1} \otimes \hat{\rho}_{j2}, \quad (2)$$

with *non-negative* p_j's, where $\hat{\rho}_{j1}$'s and $\hat{\rho}_{j2}$'s are density operators of the modes of Alice and Bob, respectively. It is evident from (2) that partial transpose operation (i.e., transpose of the density matrix with respect to only the second Hilbert space under Bob's possession), denoted PT, takes a separable density operator *necessarily* into a non-negative operator, i.e., into a bona fide density matrix. This is the Peres-Horodecki separability criterion.

In order to study the partial transpose operation in the Wigner picture, it is convenient to arrange the phase space variables and the Hermitian canonical operators into four-dimensional column vectors

$$\xi = (q_1 \quad p_1 \quad q_2 \quad p_2), \qquad \hat{\xi} = (\hat{q}_1 \quad \hat{p}_1 \quad \hat{q}_2 \quad \hat{p}_2).$$

The commutation relations take the compact form [13]

$$[\hat{\xi}_\alpha, \hat{\xi}_\beta] = i\Omega_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, 3, 4;$$

$$\Omega = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3)$$

Wigner distribution and the density operator are related through the definition [12,13]

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$$W(q,p) = \pi^{-2}\int d^2q'\langle q - q'|\hat{\rho}|q + q'\rangle \times \exp(2iq'\cdot p), \qquad (4)$$

where $q = (q_1, q_2)$, $p = (p_1, p_2)$. It follows from this definition that the partial transpose operation on the bipartite density operator transcribes faithfully into the following transformation on the Wigner distribution:

$$PT: \quad W(q_1, p_1, q_2, p_2) \rightarrow W(q_1, p_1, q_2, -p_2). \qquad (5)$$

This corresponds to a mirror reflection or "local time reversal" which inverts only the p_2 coordinate,

$$PT: \quad \xi \rightarrow \Lambda\xi, \qquad \Lambda = \mathrm{diag}(1,1,1,-1).$$

And the Peres-Horodecki separability criterion reads as follows: *if $\hat{\rho}$ is separable, then its Wigner distribution necessarily goes over into a Wigner distribution under the phase space mirror reflection Λ.* $W(\Lambda\xi)$, like $W(\xi)$, should possess the "Wigner quality," for any separable bipartite state. Roughly speaking, local time reversal, defined by Λ as above, is a symmetry in the subspace of separable states.

The Peres-Horodecki criterion has important implications for the uncertainties or second moments. Given a bipartite density operator $\hat{\rho}$, let us define $\Delta\hat{\xi} = \hat{\xi} - \langle\hat{\xi}\rangle$, where $\langle\hat{\xi}_\alpha\rangle = \mathrm{tr}\hat{\xi}_\alpha\hat{\rho}$. The four components of $\Delta\hat{\xi}$ obey the same commutation relations as $\hat{\xi}$. Similarly, we define $\Delta\xi_\alpha = \xi_\alpha - \langle\xi_\alpha\rangle$ where $\langle\xi_\alpha\rangle$ is average with respect to the Wigner distribution $W(\xi)$, and it equals $\langle\hat{\xi}_\alpha\rangle$. The uncertainties are defined as the expectations of the Hermitian operators $\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\} = (\Delta\hat{\xi}_\alpha\Delta\hat{\xi}_\beta + \Delta\hat{\xi}_\beta\Delta\hat{\xi}_\alpha)/2$:

$$\langle\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\}\rangle = \mathrm{tr}(\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\}\hat{\rho}) = \int d^4\xi\, \Delta\xi_\alpha\Delta\xi_\beta W(\xi). \qquad (6)$$

Let us now arrange the uncertainties or variances into a 4×4 real variance matrix V, defined through $V_{\alpha\beta} = \langle\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\}\rangle$. Then we have the following compact statement of the *uncertainty principle* [13]:

$$V + \frac{i}{2}\Omega \geq 0. \qquad (7)$$

Note that (7) implies, in particular, that $V > 0$.

The uncertainty principle (7) is a direct consequence of the commutation relation (3) and the non-negativity of $\hat{\rho}$. It is equivalent to the statement that $\hat{Q} = \hat{\eta}\hat{\eta}^\dagger$, with $\hat{\eta} = c_1\hat{\xi}_1 + c_2\hat{\xi}_2 + c_3\hat{\xi}_3 + c_4\hat{\xi}_4$, is nonnegative for every set of complex coefficients c_α, and hence $\langle\hat{Q}\rangle = \mathrm{tr}(\hat{Q}\hat{\rho}) \geq 0$. Viewed somewhat differently, it is *equivalent* to the statement that for every pair of real four-vectors d, d' the Hermitian operators $\hat{X}(d) = d^T\hat{\xi} = d_1\hat{q}_1 + d_2\hat{p}_1 + d_3\hat{q}_2 + d_4\hat{p}_2$ and $\hat{X}(d') = d'^T\hat{\xi} = d'_1\hat{q}_1 + d'_2\hat{p}_1 + d'_3\hat{q}_2 + d'_4\hat{p}_2$ obey

$$\langle(\Delta\hat{X}(d))^2\rangle + \langle(\Delta\hat{X}(d'))^2\rangle \geq |d'^T\Omega d| = |d_1d'_2 - d_2d'_1 + d_3d'_4 - d_4d'_3|. \qquad (8)$$

The right hand side equals $|[\hat{X}(d), \hat{X}(d')]|$. Under the Peres-Horodecki partial transpose the Wigner distribution undergoes mirror reflection, and it follows from (8) that the variances are changed to $V \rightarrow \tilde{V} = \Lambda V\Lambda$. Since $W(\Lambda\xi)$ has to be a Wigner distribution if the state under consideration is separable, we have

$$\tilde{V} + \frac{i}{2}\Omega \geq 0, \qquad \tilde{V} = \Lambda V\Lambda, \qquad (9)$$

as a *necessary* condition for separability. We may write it also in the equivalent form

$$V + \frac{i}{2}\tilde{\Omega} \geq 0, \qquad \tilde{\Omega} = \Lambda\Omega\Lambda = \begin{pmatrix} J & 0 \\ 0 & -J\end{pmatrix}, \qquad (10)$$

so that separability of $\hat{\rho}$ implies an additional restriction that has the same form as (8), with $|d'^T\Omega d|$ on the right hand side replaced by $|d'^T\tilde{\Omega}d|$. Combined with (8), this restriction reads

$$\langle(\Delta\hat{X}(d))^2\rangle + \langle(\Delta\hat{X}(d'))^2\rangle \geq |d_1d'_2 - d_2d'_1| + |d_3d'_4 - d_4d'_3|, \quad \forall\, d, d'. \qquad (11)$$

This restriction, to be obeyed by all separable states, is generically stronger than the usual uncertainty principle (8). For instance, let $\hat{X}(d)$ commute with $\hat{X}(d')$; i.e., let $d'^T\Omega d = 0$. If the state is separable, then $\hat{X}(d)$ and $\hat{X}(d')$ cannot both have arbitrarily small uncertainties unless $d'^T\tilde{\Omega}d = 0$ as well, i.e., unless $d_1d'_2 - d_2d'_1 = 0 = d_3d'_4 - d_4d'_3$. As an example, $\hat{X} = \hat{x}_1 + \hat{p}_1 + \hat{x}_2 + \hat{p}_2$ and $\hat{Y} = \hat{x}_1 - \hat{p}_1 - \hat{x}_2 + \hat{p}_2$ commute, but the sum of their uncertainties in any separable state is ≥ 4.

The Peres-Horodecki condition (11) can be simplified. Real linear canonical transformations of a two-mode system constitute the ten-parameter real symplectic group $\mathrm{Sp}(4,R)$. For every real 4×4 matrix $S \in \mathrm{Sp}(4,R)$, the irreducible canonical Hermitian operators $\hat{\xi}$ transform among themselves, leaving the fundamental commutation relation (3) invariant:

$$S \in \mathrm{Sp}(4,R): \qquad S\Omega S^T = \Omega,$$
$$\hat{\xi} \rightarrow \hat{\xi}' = S\hat{\xi}, \qquad [\hat{\xi}'_\alpha, \hat{\xi}'_\beta] = i\Omega_{\alpha\beta}. \qquad (12)$$

The symplectic group acts unitarily and irreducibly on the two-mode Hilbert space [14]. Let $U(S)$ represent the (infinite dimensional) unitary operator corresponding to $S \in \mathrm{Sp}(4,R)$. It transforms the bipartite state vector $|\psi\rangle$ to $|\psi'\rangle = U(S)|\psi\rangle$, and hence the density operator $\hat{\rho}$ to $\hat{\rho}' = U(S)\hat{\rho}U(S)^\dagger$. This transformation takes a strikingly simple form in the Wigner description, and this is one reason for the effectiveness of the Wigner picture in handling canonical transformations:

$$S: \qquad \hat{\rho} \rightarrow U(S)\hat{\rho}U(S)^\dagger \Longleftrightarrow W(\xi) \rightarrow W(S^{-1}\xi). \qquad (13)$$

The bipartite Wigner distribution simply transforms as a scalar field under $\mathrm{Sp}(4,R)$. It follows from (6) that the variance matrix transforms in the following manner:

$$S \in \mathrm{Sp}(4,R): \qquad V \rightarrow V' = SVS^T. \qquad (14)$$

The uncertainty relation (7) has an Sp$(4,R)$ invariant form (recall $S\Omega S^T = \Omega$). But separable states have to respect not just (7), but also the restriction (9), and this requirement is preserved only under the six-parameter Sp$(2,R) \otimes$ Sp$(2,R)$ subgroup of Sp$(4,R)$ corresponding to independent *local linear canonical transformations* on the subsystems of Alice and Bob:

$$S_{\rm local} \in \mathrm{Sp}(2,R) \otimes \mathrm{Sp}(2,R) \subset \mathrm{Sp}(4,R):$$

$$S_{\rm local} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \qquad S_1 J S_1^T = J = S_2 J S_2^T. \quad (15)$$

It is desirable to cast the Peres-Horodecki condition (11) in an Sp$(2,R) \otimes$ Sp$(2,R)$ invariant form. To this end, let us write the variance matrix V in the block form

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \quad (16)$$

The physical condition (7) implies $A \geq 1/4, B \geq 1/4$. As can be seen from (14), the local group changes the blocks of V in the following manner:

$$A \to S_1 A S_1^T, \qquad B \to S_2 B S_2^T, \qquad C \to S_1 C S_2^T.$$

Thus, the Sp$(2,R) \otimes$ Sp$(2,R)$ invariants associated with V are $I_1 = \det A$, $I_2 = \det B$, $I_3 = \det C$, and $I_4 = \mathrm{tr} AJCJBJC^TJ$ ($\det V$ is an obvious invariant, but it is a function of the I_k's, namely, $\det V = I_1 I_2 + I_3^2 - I_4$).

We claim that the uncertainty principle (7) is equivalent to the Sp$(2,R) \otimes$ Sp$(2,R)$ invariant statement

$$\det A \det B + \left(\frac{1}{4} - \det C\right)^2 - \mathrm{tr}(AJCJBJC^TJ) \geq \frac{1}{4}(\det A + \det B). \quad (17)$$

To prove this result, first note that (7) and (17) are equivalent for variance matrices of the special form

$$V_0 = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix}. \quad (18)$$

But any variance matrix can be brought to this special form by effecting a suitable local canonical transformation corresponding to some element of Sp$(2,R) \times$ Sp$(2,R)$. In view of the manifest Sp$(2,R) \otimes$ Sp$(2,R)$ invariant structure of (17), it follows that (7) and (17) are indeed equivalent for all variance matrices.

Under the Peres-Horodecki partial transpose or mirror reflection, we have $V \to \tilde{V} = \Lambda V \Lambda$. That is, $C \to C\sigma_3$ and $B \to \sigma_3 B \sigma_3$, while A remains unchanged [σ_3 is the diagonal Pauli matrix: $\sigma_3 = \mathrm{diag}(1,-1)$]. As a consequence, $I_3 = \det C$ flips signature while I_1, I_2, and I_4 remain unchanged. Thus, condition (9) for $\tilde{V}$ takes a form identical to (17) with only the signature in front of $\det C$ in the second term on the left hand side reversed. Thus the requirement that the variance matrix of a separable state has to obey (9), in addition to the fundamental uncertainty principle (7), takes the form

$$\det A \det B + \left(\frac{1}{4} - |\det C|\right)^2 - \mathrm{tr}(AJCJBJC^TJ) \geq \frac{1}{4}(\det A + \det B). \quad (19)$$

This is the final form of our necessary condition on the variance matrix of a separable bipartite state. This condition is invariant not only under Sp$(2,R) \otimes$ Sp$(2,R)$ *but also under mirror reflection, as it should be. It constitutes a complete description of the implication the Peres-Horodecki criterion has for the second moments.*

To summarize, conditions (7), (8), and (17) are equivalent statements of the fundamental uncertainty principle, and hence will be satisfied by every physical state. The mutually equivalent statements (9), (11), and (19) constitute the Peres-Horodecki criterion at the level of the second moments, and should necessarily be satisfied by every separable state. Interestingly, states with $\det C \geq 0$ definitely satisfy (19), which in this case is subsumed by the physical condition (17).

For the standard form V_0, our condition (19) reads

$$4(ab - c_1^2)(ab - c_2^2) \geq (a^2 + b^2) + 2|c_1 c_2| - 1/4.$$

But the point is that the separability check (19) can be applied directly on V, with no need to go to the form V_0.

We will now apply these results to Gaussian states. The mean values $\langle \hat{\xi}_\alpha \rangle$ can be changed at will using local unitary displacement operators, and so assume without loss of generality $\langle \hat{\xi}_\alpha \rangle = 0$. A (zero-mean) Gaussian state is fully characterized by its second moments, as seen from the nature of the Wigner distribution

$$W(\xi) = (4\pi^2 \sqrt{\det V})^{-1} \exp\left(-\frac{1}{2}\xi^T V^{-1} \xi\right).$$

Theorem: *The Peres-Horodecki criterion (19) is a necessary and sufficient condition for separability, for all bipartite Gaussian states.*

We begin by noting, in view of the P representation

$$\hat{\rho} = \int d^2 z_1\, d^2 z_2\, P(z_1, z_2)\, |z_1\rangle\langle z_1| \otimes |z_2\rangle\langle z_2|,$$

that a state which is classical in the quantum optics sense [non-negative $P(z_1,z_2)$] is separable. Since the local group Sp$(2,R) \otimes$ Sp$(2,R)$ does not affect separability, any Sp$(2,R) \otimes$ Sp$(2,R)$ transform of a classical state is separable too. Finally, a Gaussian state is classical if and only if $V - \frac{1}{2} \geq 0$. We will first prove a pretty little result.

Lemma: *Gaussian states with* $\det C \geq 0$ *are separable*. First consider the case $\det C > 0$. We can arrange $a \geq b$, $c_1 \geq c_2 > 0$ in the special form V_0 in (18). Let us do a local canonical transformation $S_{\rm local} = \mathrm{diag}(x, x^{-1}, x^{-1}, x)$, corresponding to reciprocal local scalings (squeezings) at the Alice and Bob ends, and follow it by $S'_{\rm local} = {\rm diag}(y, y^{-1}, y, y^{-1})$, corresponding to common local scalings at these ends. We have

$$V_0 \to V'_0 = \begin{pmatrix} y^2x^2a & 0 & y^2c_1 & 0 \\ 0 & y^{-2}x^{-2}a & 0 & y^{-2}c_2 \\ y^2c_1 & 0 & y^2x^{-2}b & 0 \\ 0 & y^{-2}c_2 & 0 & y^{-2}x^2b \end{pmatrix}.$$

Choose x such that $c_1/(x^2a - x^{-2}b) = c_2/(x^{-2}a - x^2b)$. That is, $x = [(c_1a + c_2b)/(c_2a + c_1b)]^{1/4}$. Then V'_0 acquires such a structure that it can be diagonalized by rotation through *equal* amounts in the q_1, q_2 and p_1, p_2 planes:

$$V'_0 \to V''_0 = {\rm diag}(\kappa_+, \kappa'_+, \kappa_-, \kappa'_-);$$

$$\kappa_\pm = \frac{1}{2} y^2\{x^2a + x^{-2}b \pm [(x^2a - x^{-2}b)^2 + 4c_1^2]^{1/2}\},$$

$$\kappa'_\pm = \frac{1}{2} y^{-2}\{x^{-2}a + x^2b \pm [(x^{-2}a - x^2b)^2 + 4c_2^2]^{1/2}\}.$$

Such an equal rotation is a canonical transformation; it preserves the uncertainty principle, since it is canonical, and the pointwise non-negativity of the P distribution, since it is a rotation. For our diagonal V''_0, the uncertainty principle $V''_0 + \frac{i}{2}\Omega \geq 0$ simply reads that the product $\kappa_-\kappa'_- \geq 1/4$. It follows that we can choose y such that $\kappa_-, \kappa'_- = 1/2$ (for instance, choose y such that $\kappa_- = \kappa'_-$), i.e., $V''_0 \geq 1/2$. Since V'_0 and V''_0 are rotationally related, this implies $V'_0 \geq 1/2$, and hence V'_0 corresponds to positive P distribution or separable state. This in turn implies that the original V corresponds to a separable state, since V and V'_0 are related by local transformation. This completes proof for the case det$C > 0$.

Now suppose det$C = 0$, so that in V_0 we have $c_1 \geq 0 = c_2$. Carry out a local scaling corresponding to $S_{\rm local} = {\rm diag}(\sqrt{2a}, 1/\sqrt{2a}, \sqrt{2b}, 1/\sqrt{2b})$, taking $V_0 \to V'_0$; the diagonal entries of V'_0 are $(2a^2, 1/2, 2b^2, 1/2)$, and the two nonzero off-diagonal entries equal $2abc_1$. With this form for V'_0, the uncertainty principle $V'_0 + \frac{i}{2}\Omega \geq 0$ implies $V'_0 \geq 1/2$, establishing separability of the Gaussian state. This completes proof of our lemma.

Proof of the main theorem is completed as follows. We consider in turn the two distinct cases det$C < 0$ and det$C \geq 0$. Suppose det$C < 0$. Then there are two possibilities. If (19) is violated, then the Gaussian state is definitely entangled since (19) is a necessary condition for separability. If (19) is respected, then the mirror reflected state is a physical Gaussian state with det$C > 0$ (recall that mirror reflection flips the signature of det$C$), and is separable by the above lemma. This implies separability of the original state, since a mirror reflected separable state is separable. Finally, suppose det$C \geq 0$. Condition (19) is definitely satisfied since it is subsumed by the uncertainty principle (17) in this case. By our lemma, the state is separable. This completes proof of the theorem.

We used the scaled commutation relation $[\hat{q}, \hat{p}] = i$. Reinserting a scale parameter $m \geq 0$, this relation becomes $[\hat{q}, \hat{p}] = m^2 i$, the inequality (7) becomes $V + \frac{i}{2}m^2\Omega \geq 0$, and the $1/4$ on the left and right hand sides of (17) and (19) gets replaced by $m^2/4$.

Finally, our analysis has been presented in the Wigner picture. But the geometric interpretation of the partial transpose as mirror reflection in phase space holds for the other (s-ordered) quasiprobabilities as well.

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Note added.—Since completion of this work, a preprint by Duan *et al.* [15] describing an interesting approach to separability has appeared. One would have expected the insufficiency of the Peres-Horodecki criterion for separability to grow with dimension, rendering this criterion of little value in the limit of infinite dimension. Therefore these authors aim at an inseparability condition independent of the Peres-Horodecki criterion. But our approach simply exploits the geometric flavor the Peres-Horodecki partial transpose criterion acquires in this infinite limit.

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