Inseparability Criterion for Continuous Variable Systems

Lu-Ming Duan,[1,2,*] G. Giedke,[1] J. I. Cirac,[1] and P. Zoller[1]

[1]*Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria*

[2]*Laboratory of Quantum Communication and Quantum Computation, University of Science and Technology of China, Hefei 230026, China*

(Received 17 August 1999)

An inseparability criterion based on the total variance of a pair of Einstein-Podolsky-Rosen type operators is proposed for continuous variable systems. The criterion provides a sufficient condition for entanglement of any two-party continuous variable states. Furthermore, for all Gaussian states, this criterion turns out to be a necessary and sufficient condition for inseparability.

PACS numbers: 03.67.–a, 03.65.Bz, 42.50.Dv, 89.70.+c

It is now believed that quantum entanglement plays an essential role in all branches of quantum information theory [1]. A problem of great importance is then to check if a state, generally mixed, is entangled or not. Concerning this problem, Peres proposed an inseparability criterion based on partial transpose of the composite density operator [2], which provides a sufficient condition for entanglement. This criterion was later shown by Horodecki to be a necessary and sufficient condition for inseparability of the (2×2)- or (2×3)-dimensional states, but not to be necessary any more for higher-dimensional states [3,4]. Many recent protocols for quantum communication and computation are based on continuous variable quantum systems [5–11], and the continuous variable optical system has been used to experimentally realize the unconditional quantum teleportation [12]. Hence, it is desirable to know if a continuous variable state is entangled or not.

In this paper, we propose a simple inseparability criterion for continuous variable states. The criterion is based on the calculation of the total variance of a pair of Einstein-Podolsky-Rosen (EPR) type operators. We find that, for any separable continuous variable states, the total variance is bounded from below by a certain value resulting from the uncertainty relation, whereas for entangled states this bound can be exceeded. So, violation of this bound provides a sufficient condition for inseparability of the state. We then investigate how strong the bound is for the set of Gaussian states, which is of great practical importance. It is shown that for a Gaussian state, the compliance with the low bound by a certain pair of EPR type operators guarantees that the state has a P representation with positive distribution, so the state must be separable. Hence we obtain a necessary and sufficient inseparability criterion for all of the Gaussian continuous variable states.

We say a quantum state ρ of two modes 1 and 2 is separable if, and only if, it can be expressed in the following form:

$$\rho = \sum_i p_i \rho_{i1} \otimes \rho_{i2}, \tag{1}$$

where we assume ρ_{i1} and ρ_{i2} to be normalized states of the modes 1 and 2, respectively, and $p_i \geq 0$ to satisfy $\sum_i p_i = 1$.

A maximally entangled continuous variable state can be expressed as a co-eigenstate of a pair of EPR type operators [13], such as $\hat{x}_1 + \hat{x}_2$ and $\hat{p}_1 - \hat{p}_2$. Therefore, the total variance of these two operators reduces to zero for maximally entangled continuous variable states. Of course, the maximally entangled continuous variable states are not physical, but for the physically entangled continuous variable states—the two-mode squeezed states [14]—this variance will rapidly tend to zero by increasing the degree of squeezing. Interestingly, we find that, for any separable state, there exists a lower bound to the total variance. To be more general, we consider the following type of EPR-like operators:

$$\hat{u} = |a|\hat{x}_1 + \frac{1}{a}\hat{x}_2, \tag{2a}$$

$$\hat{v} = |a|\hat{p}_1 - \frac{1}{a}\hat{p}_2, \tag{2b}$$

where we assume a is an arbitrary (nonzero) real number. For any separable state, the total variance of any pair of EPR-like operators in the form of Eqs. (2a) and (2b) should satisfy a lower bound indicated by the following theorem:

Theorem 1.–Sufficient criterion for inseparability: For any separable quantum state ρ, the total variance of a pair of EPR-like operators defined by Eqs. (2a) and (2b) with the commutators $[\hat{x}_j, \hat{p}_{j'}] = i\delta_{jj'}$ $(j, j' = 1, 2)$ satisfies the inequality

$$\langle(\Delta\hat{u})^2\rangle_\rho + \langle(\Delta\hat{v})^2\rangle_\rho \geq a^2 + \frac{1}{a^2}. \tag{3}$$

Proof.—We can directly calculate the total variance of the $\hat{u}$ and $\hat{v}$ operators using the decomposition (1) of the density operator ρ, and finally get the following expression:

$$\begin{aligned}
\langle(\Delta\hat{u})^2\rangle_\rho + \langle(\Delta\hat{v})^2\rangle_\rho &= \sum_i p_i(\langle\hat{u}^2\rangle_i + \langle\hat{v}^2\rangle_i) - \langle\hat{u}\rangle_\rho^2 - \langle\hat{v}\rangle_\rho^2 \\
&= \sum_i p_i\left(a^2\langle\hat{x}_1^2\rangle_i + \frac{1}{a^2}\langle\hat{x}_2^2\rangle_i + a^2\langle\hat{p}_1^2\rangle_i + \frac{1}{a^2}\langle\hat{p}_2^2\rangle_i\right) \\
&\quad + 2\frac{a}{|a|}\left(\sum_i p_i\langle\hat{x}_1\rangle_i\langle\hat{x}_2\rangle_i - \sum_i p_i\langle\hat{p}_1\rangle_i\langle\hat{p}_2\rangle_i\right) - \langle\hat{u}\rangle_\rho^2 - \langle\hat{v}\rangle_\rho^2 \\
&= \sum_i p_i\left(a^2\langle(\Delta\hat{x}_1)^2\rangle_i + \frac{1}{a^2}\langle(\Delta\hat{x}_2)^2\rangle_i + a^2\langle(\Delta\hat{p}_1)^2\rangle_i + \frac{1}{a^2}\langle(\Delta\hat{p}_2)^2\rangle_i\right) \\
&\quad + \sum_i p_i\langle\hat{u}\rangle_i^2 - \left(\sum_i p_i\langle\hat{u}\rangle_i\right)^2 + \sum_i p_i\langle\hat{v}\rangle_i^2 - \left(\sum_i p_i\langle\hat{v}\rangle_i\right)^2. \qquad (4)
\end{aligned}$$

In Eq. (4), the symbol $\langle\cdots\rangle_i$ denotes the average over the product density operator $\rho_{i1}\otimes\rho_{i2}$. It follows from the uncertainty relation that $\langle(\Delta\hat{x}_j)^2\rangle_i + \langle(\Delta\hat{p}_j)^2\rangle_i \geq |[\hat{x}_j,\hat{p}_j]| = 1$ for $j = 1,2$, and, moreover, by applying the Cauchy-Schwarz inequality $(\sum_i p_i)(\sum_i p_i\langle\hat{u}\rangle_i^2) \geq (\sum_i p_i|\langle\hat{u}\rangle_i|)^2$, we know that the last line of Eq. (4) is bounded from below by zero. Hence, the total variance of the two EPR-like operators $\hat{u}$ and $\hat{v}$ is bounded from below by $a^2 + \frac{1}{a^2}$ for any separable state. This completes the proof of the theorem.

Note that this theorem in fact gives a set of inequalities for separable states. The operators $\hat{x}_j, \hat{p}_j$ $(j = 1,2)$ in the definition (1) can be any local operators satisfying the commutators $[\hat{x}_j,\hat{p}_{j'}] = i\delta_{jj'}$. In particular, if we apply an arbitrary local unitary operation $U_1\otimes U_2$ to the operators $\hat{u}$ and $\hat{v}$, the inequality (3) remains unchanged. Note also that without loss of generality we have taken the operators x_j and p_j dimensionless.

For inseparable states, the total variance of the $\hat{u}$ and $\hat{v}$ operators is required by the uncertainty relation to be larger than or equal to $|a^2 - \frac{1}{a^2}|$, which reduces to zero for $a = 1$. For separable states the much stronger bound given by Eq. (3) must be satisfied. A natural question is then how strong is the bound. Is it strong enough to ensure that, if some inequality in the form of Eq. (3) is satisfied, the state necessarily becomes separable? Of course, it will be very difficult to consider this problem for arbitrary continuous variable states. However, in recent experiments and protocols for quantum communication [5–12], continuous variable entanglement is generated by two-mode squeezing or by beam splitters, and the communication noise results from photon absorption and thermal photon emission. All of these processes lead to Gaussian states. So, we will limit ourselves to consider Gaussian states, which are of great practical importance. We find that the inequality (3) indeed gives a necessary and sufficient inseparability criterion for all of the Gaussian states. To present and prove our main theorem, we need first mention some notations and results for Gaussian states.

It is convenient to represent a Gaussian state by its Wigner characteristic function. A two-mode state with the density operator ρ has the following Wigner characteristic function [14]:

$$\begin{aligned}
\chi^{(w)}(\lambda_1,\lambda_2) &= \mathrm{tr}[\rho\exp(\lambda_1\hat{a}_1 - \lambda_1^*\hat{a}_1^\dagger + \lambda_2\hat{a}_2 - \lambda_2^*\hat{a}_2^\dagger)] \\
&= \mathrm{tr}\{\rho\exp[i\sqrt{2}(\lambda_1^I\hat{x}_1 + \lambda_1^R p_1 + \lambda_2^I\hat{x}_2 \\
&\qquad + \lambda_2^R\hat{p}_2)]\}, \qquad (5)
\end{aligned}$$

where the parameters $\lambda_j = \lambda_j^R + i\lambda_j^I$, and the annihilation operators $\hat{a}_j = \frac{1}{\sqrt{2}}(\hat{x}_j + i\hat{p}_j)$, with the quadrature amplitudes $\hat{x}_j, \hat{p}_j$ satisfying the commutators $[\hat{x}_j,\hat{p}_{j'}] = i\delta_{jj'}$ $(j,j' = 1,2)$. For a Gaussian state, the Wigner characteristic function $\chi^{(w)}(\lambda_1,\lambda_2)$ is a Gaussian function of λ_j^R and λ_j^I [14]. Without loss of generality, we can write $\chi^{(w)}(\lambda_1,\lambda_2)$ in the form

$$\chi^{(w)}(\lambda_1,\lambda_2) = \exp\left[-\frac{1}{2}(\lambda_1^I,\lambda_1^R,\lambda_2^I,\lambda_2^R)M(\lambda_1^I,\lambda_1^R,\lambda_2^I,\lambda_2^R)^T\right]. \qquad (6)$$

In Eq. (6), linear terms in the exponent are not included since they can be easily removed by some local displacements of $\hat{x}_j, \hat{p}_j$ and thus have no influence on separability or inseparability of the state. The correlation property of the Gaussian state is completely determined by the 4×4 real symmetric correlation matrix M, which can be expressed as

$$M = \begin{pmatrix} G_1 & C \\ C^T\mathrm{pt} & G_2 \end{pmatrix}, \qquad (7)$$

where G_1, G_2, and C are 2×2 real matrices. To study the separability property, it is convenient to first transform the Gaussian state to some standard forms through local linear unitary Bogoliubov operations (LLUBOs) $U_l = U_1\otimes U_2$. In the Heisenberg picture, the general form of the LLUBO U_l is expressed as $U_l(\hat{x}_j,\hat{p}_j)^T U_l^\dagger = H_j(\hat{x}_j,\hat{p}_j)^T$ for $j = 1,2$, where H_j is some 2×2 real matrix with $\det H_j = 1$. Any LLUBO is obtainable by combining the squeezing transformation together with some rotations [15]. We have the following two lemmas concerning the standard forms of the Gaussian state.

Lemma 1.—Standard form I: Any Gaussian state ρ_G can be transformed through LLUBOs to the standard form I with the correlation matrix given by

$$M_s^{\mathrm{I}} = \begin{pmatrix} n & & c & \\ & n & & c' \\ c & & m & \\ & c' & & m \end{pmatrix}, \qquad (n, m \geq 1). \quad (8)$$

Proof.—A LLUBO on the state ρ_G transforms the correlation matrix M in the Wigner characteristic function in the following way:

$$\begin{pmatrix} V_1 & \\ & V_2 \end{pmatrix} M \begin{pmatrix} V_1^T & \\ & V_2^T \end{pmatrix}, \quad (9)$$

where V_1 and V_2 are real matrices with $\det V_1 = \det V_2 = 1$. Since the matrices G_1 and G_2 in Eq. (7) are real symmetric, we can choose first a LLUBO with orthogonal V_1 and V_2 which diagonalize G_1 and G_2, and then a local squeezing operation which transforms the diagonalized G_1 and G_2 into the matrices $G_1' = nI_2$ and $G_2' = mI_2$, respectively, where I_2 is the 2×2 unit matrix. After these two steps of operations, we assume that the matrix C in Eq. (7) is changed into C', which always has a singular value decomposition; thus it can be diagonalized by another LLUBO with suitable orthogonal V_1 and V_2. The last orthogonal LLUBO no longer influences G_1' and G_2' since they are proportional to the unit matrix. Hence, any Gaussian state can be transformed by three-step LLUBOs to the standard form I. The four parameters n, m, c, and c' in the standard form I are related to the four invariants $\det G_1$, $\det G_2$, $\det C$, and $\det M$ of the correlation matrix under LLUBOs by the equations $\det G_1 = n^2$, $\det G_2 = m^2$, $\det C = cc'$, and $\det M = (nm - c^2)(nm - c'^2)$, respectively.

Lemma 2.—Standard form II: Any Gaussian state ρ_G can be transformed through LLUBOs into the standard form II with the correlation matrix given by

$$M_s^{\mathrm{II}} = \begin{pmatrix} n_1 & & c_1 & \\ & n_2 & & c_2 \\ c_1 & & m_1 & \\ & c_2 & & m_2 \end{pmatrix}, \quad (10)$$

where the n_i, m_i, and c_i satisfy

$$\frac{n_1 - 1}{m_1 - 1} = \frac{n_2 - 1}{m_2 - 1}, \quad (11a)$$

$$|c_1| - |c_2| = \sqrt{(n_1 - 1)(m_1 - 1)} - \sqrt{(n_2 - 1)(m_2 - 1)}. \quad (11b)$$

Proof.—Any Gaussian state can be tranformed through LLUBOs to the standard form I. We then apply two additional local squeezing operations on the standard form I, and get the state with the following correlation matrix:

$$M' = \begin{pmatrix} nr_1 & & \sqrt{r_1 r_2}\, c & \\ & \frac{n}{r_1} & & \frac{c'}{\sqrt{r_1 r_2}} \\ \sqrt{r_1 r_2}\, c & & mr_2 & \\ & \frac{c'}{\sqrt{r_1 r_2}} & & \frac{m}{r_2} \end{pmatrix}, \quad (12)$$

where r_1 and r_2 are arbitrary squeezing parameters. M' in Eq. (12) has the standard form M_s^{II} (10) if r_1 and r_2 satisfy the following two equations:

$$\frac{\frac{n}{r_1} - 1}{nr_1 - 1} = \frac{\frac{m}{r_2} - 1}{mr_2 - 1}, \quad (13)$$

$$\sqrt{r_1 r_2}\,|c| - \frac{|c'|}{\sqrt{r_1 r_2}} = \sqrt{(nr_1 - 1)(mr_2 - 1)} - \sqrt{\left(\frac{n}{r_1} - 1\right)\left(\frac{m}{r_2} - 1\right)}. \quad (14)$$

Our task remains to prove that Eqs. (13) and (14) are indeed satisfied by some positive r_1 and r_2 for arbitrary Gaussian states. Without loss of generality, we assume $|c| \geq |c'|$ and $n \geq m$. From Eq. (13), r_2 can be expressed as a continuous function of r_1 with $r_2(r_1 = 1) = 1$ and $r_2(r_1) \xrightarrow{r_1 \to \infty} m$. Substituting this expression $r_2(r_1)$ into Eq. (14), we construct a function $f(r_1)$ by subtracting the right-hand side of Eq. (14) from the left-hand side, i.e., $f(r_1) = \text{left}(14) - \text{right}(14)$. Obviously, $f(r_1 = 1) = |c| - |c'| \geq 0$, and $f(r_1) \xrightarrow{r_1 \to \infty} \sqrt{r_1 m}\,[|c| - \sqrt{n(m - \frac{1}{m})}\,] \leq 0$, where the inequality $|c| \leq \sqrt{n(m - \frac{1}{m})}$ results from the physical condition $\langle(\Delta \hat{u}_0)^2\rangle + \langle(\Delta \hat{v}_0)^2\rangle \geq |[\hat{u}_0, \hat{v}_0]|$ with $\hat{u}_0 = \sqrt{m - \frac{1}{m}}\,\hat{x}_1 - \frac{c}{|c|}\sqrt{n}\,\hat{x}_2$ and $\hat{v}_0 = \frac{\sqrt{n}}{m}\,\hat{p}_2$. It follows from continuity that there must exist a $r_1^* \in [1, \infty)$ which makes $f(r_1 = r_1^*) = 0$. Therefore Eqs. (13) and (14) have at least one solution. This proves lemma 2.

We remark that, corresponding to a given standard form I or II, there is a class of Gaussian states which is equivalent under LLUBOs. Note that separability or inseparability is a property not influenced by LLUBOs, so all of the Gaussian states with the same standard forms have the same separability or inseparability property. With the above preparations, we now present the following main theorem:

Theorem 2.—Necessary and sufficient inseparability criterion for Gaussian states: A Gaussian state ρ_G is separable if, and only if, when expressed in its standard form II, the inequality (3) is satisfied by the following two EPR type operators

$$\hat{u} = a_0 \hat{x}_1 - \frac{c_1}{|c_1|}\frac{1}{a_0}\hat{x}_2, \quad (15a)$$

$$\hat{v} = a_0 \hat{p}_1 - \frac{c_2}{|c_2|}\frac{1}{a_0}\hat{p}_2, \quad (15b)$$

where $a_0^2 = \sqrt{\frac{m_1-1}{n_1-1}} = \sqrt{\frac{m_2-1}{n_2-1}}$.

Proof.—The "only if" part follows directly from theorem 1. We only need to prove the "if" part. From lemma 2, we can first transform the Gaussian state through LLUBOs to the standard form II. The state after transformation is denoted by ρ_G^{II}. Then, substituting the expressions (15a) and (15b) of $\hat{u}$ and $\hat{v}$ into the inequality (3), and calculating $\langle(\Delta\hat{u})^2\rangle + \langle(\Delta\hat{v})^2\rangle$ by using the correlation matrix M_s^{II}, we get the following inequality:

$$a_0^2 \frac{n_1 + n_2}{2} + \frac{m_1 + m_2}{2a_0^2} - |c_1| - |c_2| \geq a_0^2 + \frac{1}{a_0^2}, \tag{16}$$

which, combined with Eqs. (11), yields

$$|c_1| \leq \sqrt{(n_1 - 1)(m_1 - 1)}, \tag{17a}$$

$$|c_2| \leq \sqrt{(n_2 - 1)(m_2 - 1)}. \tag{17b}$$

The inequalities (17a) and (17b) ensures that the matrix $M_s^{\mathrm{II}} - I$ is positive semidefinite. So there exists a Fourier transformation to the following normal characteristic function of the state ρ_G^{II}:

$$\begin{aligned}\chi_{\mathrm{II}}^{(n)}(\lambda_1, \lambda_2) &= \chi_{\mathrm{II}}^{(w)}(\lambda_1, \lambda_2) \exp\left[\frac{1}{2}(|\lambda_1|^2 + |\lambda_2|^2)\right] \\ &= \exp\left[-\frac{1}{2}(\lambda_1^I, \lambda_1^R, \lambda_2^I, \lambda_2^R)(M_s^{\mathrm{II}} - I) \right. \\ &\qquad \left. \times (\lambda_1^I, \lambda_1^R, \lambda_2^I, \lambda_2^R)^T\right]. \end{aligned} \tag{18}$$

This means that ρ_G^{II} can be expressed as

$$\rho_G^{\mathrm{II}} = \int d^2\alpha\, d^2\beta\, P(\alpha, \beta)\, |\alpha, \beta\rangle\langle\alpha, \beta|, \tag{19}$$

where $P(\alpha, \beta)$ is the Fourier transformation of $\chi_{\mathrm{II}}^{(n)}(\lambda_1, \lambda_2)$ and thus is a positive Gaussian function. Equation (19) shows ρ_G^{II} is separable. Since the original Gaussian state ρ_G differs from ρ_G^{II} by only some LLUBOs, it must also be separable. This completes the proof of theorem 2.

Now we have a necessary and sufficient inseparability criterion for all of the Gaussian states. We conclude the paper by applying this criterion to a simple example. Consider a two-mode squeezed vacuum state $e^{-r(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)}|\mathrm{vac}\rangle$ with the squeezing parameter r. This state has been used in recent experiments for continuous variable quantum teleportation [12]. Suppose that the two optical modes are subject to independent thermal noise during transmission with the same damping coefficient denoted by η and the same mean thermal photon number denoted by $\overline{n}$. It is easy to show that, after time t, the standard correlation matrix for this Gaussian state has the form of Eq. (8) with $n = m = \cosh(2r)e^{-2\eta t} + (2\overline{n} + 1)(1 - e^{-2\eta t})$ and $c = -c' = \sinh(2r)e^{-2\eta t}$ [16]. Therefore the inseparability criterion means that, if the transmission time t satisfies

$$t < \frac{1}{2\eta} \ln\left(1 + \frac{1 - e^{-2r}}{2\overline{n}}\right), \tag{20}$$

the state is entangled; otherwise it becomes separable. Interestingly, Eq. (20) shows that, if there is only vacuum fluctuation noise, i.e., $\overline{n} = 0$ (this seems to be a good approximation for optical frequency), the initial squeezed state is always entangled. This result does not remain true if thermal noise is present. In the limit $\overline{n} \gg 1$, the state is no longer entangled when the transmission time $t \geq \frac{1-e^{-2r}}{4\eta\overline{n}}$.

Note added.—After submission of this work, we became aware of a recent preprint by R. Simon (quant-ph/9909044), which shows that the Peres-Horodecki criterion also provides a necessary and sufficient condition for inseparability of Gaussian continuous variable quantum states.

This work was funded by the Austrian Science Foundation and by the European TMR Network Quantum Information. G. G. acknowledges support by the Friedrich-Naumann-Stiftung.

*Email address: luming.duan@uibk.ac.at

[1] C. H. Bennett, Phys. Today **48**, No. 10, 24 (1995); D. P. DiVincenzo, Science **270**, 255 (1995).
[2] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[3] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[4] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
[5] L. Vaidman, Phys. Rev. A **49**, 1473 (1994).
[6] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).
[7] S. L. Braunstein, Nature (London) **394**, 47 (1998).
[8] S. Lloyd and S. L. Braunstein, Phys. Rev. Lett. **82**, 1784 (1999).
[9] G. J. Milburn and S. L. Braunstein, quant-ph/9812018.
[10] P. Loock, A. L. Braunstein, and H. J. Kimble, quant-ph/9902030.
[11] A. S. Parkins and H. J. Kimble, quant-ph/9904062.
[12] A. Furusawa *et al.,* Science **282**, 706 (1998).
[13] A. Einstein, B. Podolsky, and R. Rosen, Phys. Rev. **47**, 777 (1935).
[14] C. W. Gardiner and P. Zoller, *Quantum Noise* (Springer-Verlag, Berlin, 1999), 2nd ed.
[15] S. L. Braunstein, quant-ph/9904002.
[16] L. M. Duan and G. C. Guo, Quantum Semiclass. Opt. **9**, 953 (1997).