## Phase Order in Chaotic Maps and in Coupled Map Lattices

Wei Wang,<sup>1,2</sup> Zonghua Liu,<sup>2</sup> and Bambi Hu<sup>2</sup>

<sup>1</sup>National Laboratory of Solid State Microstructure, Nanjing University, Nanjing 210093, China

<sup>2</sup>Center for Nonlinear Studies and Department of Physics, Hong Kong Baptist University, Hong Kong, China

(Received 16 August 1999)

By defining a direction phase as the direction of two sequential iterations of the logistic map, a transition of a net direction phase M from zero to a finite value as the parameter  $\mu$  increases is found. Near the transition point  $\mu_0$  a scaling  $M \sim (\mu - \mu_0)^{\alpha}$  with  $\alpha = 0.5$  is obtained. The order state of the direction phases in a coupled map lattice is also studied. A phase synchronization of the direction phases is found although the lattices still remain chaotic.

PACS numbers: 05.45.Ra, 05.45.Xt, 05.50.+q

Generally, in a dynamical system, a chaotic state means that the system follows a chaotic trajectory and shows randomlike features. For example, in its chaotic state, the logistic map shows randomlike values during iterations. However, for the bifurcation parameter  $3.0 < \mu <$ 3.6786, these values are settled into two main bands until they are merged together at  $\mu_0 = 3.6786$  [1–3]. The values of sequential iterations always periodically hop between these two bands even when the map is situated in its chaotic states. Such a periodicity actually implies a type of symmetry (two-band symmetry). A "direction phase" of the iterations, if we define it as the direction of two sequential iterations (or the trajectory), has a specific choice or an ordered arrangement, i.e., each up phase always followed by a down phase (see Fig. 1). That is, the net direction phase is zero, which is just like the spins arranged in an antiferromagnetism [4]. Obviously, the symmetry will be broken when  $\mu \geq \mu_0$  where the two bands are merged into one large band. Our question is whether the symmetry breaking (the bands merging) in the chaotic map results in a state of nonzero net direction phase or a transition from an ordered arrangement of the direction phases to a disordered one. Is such an ordered or disordered arrangement of direction phases relevant to any mechanism in the pattern formation [5] or phase synchronization [6] in the dynamical systems?

In this paper, we study the questions mentioned above. For a logistic map, we find a transition of the direction phase from an ordered arrangement to a disordered one occurring at  $\mu_0$ . For such a disordered arrangement there exists a net up direction M of the trajectory, which shows a scaling  $M \sim (\mu - \mu_0)^{\alpha}$  with  $\alpha = 0.5$  near  $\mu_0$ . This scaling and exponent  $\alpha$  relate to the intrinsic feature of the quadratic maximum of the map. For a coupled map lattice, we find that there exists a spatial order of such direction phases which show a synchronized oscillation with the iterations of the lattices when the coupling is strong. This direction-phase ordering is argued relevant to the pattern formation and phase synchronization in the chaotic system.

Let us consider the logistic map  $X_{n+1} = \mu X_n(1 - X_n)$ , where  $\mu \in [1, 4]$  and  $0 < X_n < 1$ . As  $\mu$  increases from 1 to 4, the map experiences a period doubling to chaos

[1-3]. At  $\mu = 3.18$ , the map presents chaos, and at  $\mu_0$  the two branches of chaotic bands merge together. Clearly, the map is symmetric about two branches, i.e., a two-band symmetry, when  $\mu < \mu_0$  [see Fig. 2(a)]. The values of iterations  $X_n$  always follow a "up" direction and then a "down" one, which contributes no net "up" or "down" phase [see Figs. 2(a) and 1(a)], just like the spins arranged in an antiferromagnetism. Here the net direction phase, similar to the magnetization [4], is defined as  $M(\mu) = T^{-1} \sum_{n=1}^{T} S(n)$  over a large number of itera-tions T. Here S(n) = 1 for  $X_{n+1} - X_n > 1$  and S(n) =-1 for  $X_{n+1} - X_n < 1$  are defined as the up phase  $S_{\uparrow}$ and down phase  $S_{\downarrow}$ , respectively. The zero net direction phase, i.e., M = 0, extends to  $\mu = \mu_0$ , where two chaotic bands are merged together. For  $\mu > \mu_0$ , there is a net up phase with  $M \neq 0$ , i.e., a preferential state with up phase [see Figs. 1(b) and 2(b)]. However, the arrangement of the up and down direction phases is disordered. It is noted that there are some flat plateaus with constant M relating to the periodic windows in the bifurcation diagram [see Fig. 2(b)]. In addition, there is also a scaling  $M \sim (\mu - \mu_0)^{\alpha}$  with  $\alpha = 0.5$  near the transition  $\mu_0$  [see Fig. 2(c)].

Such a disordered state of finite net direction phases results clearly from the symmetry breaking of the two-band structure. Let us define the preimage of the unstable fixed point,  $X_c$ , and the unstable fixed point itself,  $X_f$ . Then the whole interval  $[X_c, X_f]$  is mapped onto  $[X_f, X_{max}]$ ,



FIG. 1. The direction phase and its similarity to the spin and the binary representations. (a) For  $\mu = 3.65$ , the net up phase is zero; (b) for  $\mu = 3.90$ , the net up phase is not zero.

© 2000 The American Physical Society



FIG. 2. The bifurcation diagram and the net direction phase M versus  $\mu$  for the logistic map. (a) The bifurcation diagram. (b) M vs  $\mu$ ; The inset shows the case of the marked region in (a). (c) The scaling of M with  $\mu$  near the transition  $\mu_0$  (open circles) and  $\mu'_0$  (open triangles).

such that S(n) = 1 for all points in  $[X_c, X_f]$ , whereas the interval  $[X_f, X_{max}]$  is mapped to  $[X_{min}, X_f]$ , such that S(n) = -1 for all of these points. As  $\mu$  increases from below  $\mu_0$  to above  $\mu_0$ , the left end point of the support of the measure  $X_{\min}$  is shifted from the right of  $X_c$  to the left of  $X_c$  [7]. The nonzero (or zero) M depends on whether the invariant measure has (or has no) contribution on the left of  $X_c$ . Thus, for  $\mu \leq \mu_0$ , we have  $X_{\min} \geq X_c$ and M = 0. Clearly, increasing  $\mu$  leads to a linear increase of the interval of the support of the measure in the left of  $X_{c_1}$  i.e.,  $(X_{c_1} - X_{\min}) \propto (\mu - \mu_0)$ , and all of the points in  $[X_{\min}, X_c]$  lead to two successive S(n) = 1and S(n + 1) = 1 before the next single S(n + 2) = -1 follows. Therefore, we have  $M \sim \sum_{n=1}^{T} S(n)$  being proportional to the measure of the interval  $[X_{\min}, X_c]$ , i.e.,  $M \sim (\mu - \mu_0)^{\alpha}$  with  $\alpha = 1/2$  since the square root singularities of invariant measure map at the edge [3,8]. As a result, the scaling near  $\mu_0$  is an intrinsic feature of the logistic map itself. In addition, it is known that at  $\mu_0$ there is a collision between the unstable period-1 orbit and the chaotic attractor, which leads to the so-called crisis [9]. Since the birth of the crisis often induces complex dynamical behavior [9], presumably, the net direction phase could be one measure of such complex behavior.

Interestingly, the net direction phase also appears in each chaotic band as long as there exists a merging of two subchaotic bands. From the marked region in Fig. 2(a), we can see such a transition [the inset in Fig. 2(b)]. The scaling  $M \sim (\mu - \mu_0')^{\alpha}$  is the same as that near  $\mu_0$ , and the same exponent  $\alpha = 0.5$  is also obtained [see Fig. 2(c)]. Because of the self-similarity of the bifurcation diagram shown in Fig. 2(a), we may find the direction-phase transition existing in various levels. That is, in the whole bifurcation range there are many transitions at each local two-band symmetry breaking. For example, in the marked region, there is a net up phase, while its symmetric counterpart has a net down phase. As a matter of fact, the hierarchical characteristic mentioned above relates to the definition of the direction phase. The direction phase defined in Fig. 1 is  $N_s = 2^{N_B-1} = 1$  with  $N_s$  being the interval of the steps of the iterations and  $N_B$  the number of the bands after the merging. Near  $\mu_0$ , the number of the bands after merging is one,  $N_B = 1$ ; the direction phase is defined as  $X_{n+N_s} - X_n = X_{n+1} - X_n$ . Similarly, for the region marked in Fig. 2(a), the number of bands after merging is two, i.e.,  $N_B = 2$ ; then the direction phase is given by  $X_{n+N_s} - X_n = X_{n+2} - X_n$ . In addition, for the period-3 bands (for  $\mu \in [3.83, 3.86]$ ), we have  $N_s = 3 \times 2^{N_B-1}$  with  $N_B = 1$ . In general, we have  $N_s = q \times 2^{N_B-1}$  with  $q = 1, 3, 5, \ldots$ , and  $N_B = 1, 2, 4, \ldots$ , for different periods and different bands. Hence, we may conclude that, as long as there exists a symmetry breaking of a two-band structure in a chaotic map, there will always be a direction-phase transition.

Now, let us turn to the study of the collective behavior of the direction phases of a coupled map lattice. The motivation for such a study is as follows: Because of the similarity of the direction phase to the spin, every individual map in the lattices is always in a certain direction, either up or down at each time. Thus, in the presence of coupling, there may exist some correlation between the direction phases of maps, which will present nontrial-collective dynamical and statistical behaviors just like the spins in a two-dimensional Ising model. There may be some clusters (or domains) of maps with the same direction phases, implying a relation to the pattern formation [10]. In addition, since the direction phase has similar features to the phases defined in a number of studies [6,11], we expect that our system also exhibits an interesting phenomenon, the phase synchronization. The phase synchronization in such a mapping dynamical system shows a transition from an in-phase state to an antiphase one.

We consider a two-dimensional coupled map lattice,

$$X_{n+1}(i,j) = f(X_n(i,j)) + \epsilon [f(X_n(i+1,j)) + f(X_n(i-1,j)) + f(X_n(i,j+1)) + f(X_n(i,j-1)) - 4f(X_n(i,j))]/4,$$
(1)



FIG. 3. The net direction phase M versus  $\mu$  for a lattice [Eq. (1)] with different couplings  $\epsilon$ . The inset shows the onset of the transition  $\mu_c$  versus  $\epsilon$ .

where  $\epsilon$  is the coupling strength and  $f(x) = \mu x(1 - x)$ . That is, the lattices include  $L \times L$  (L = 100) logistic maps. A periodic boundary condition is used. The interaction is taken as the diffusive interaction [12].

In Fig. 3, we show the net direction phase of the lattice M versus the parameter  $\mu$  for different coupling strengths. Here M is defined as a spatiotemporal average

 $M = (T^{-1} \times L^{-2}) \sum_{n=1}^{T} \sum_{i,j} S^{(i,j)}(n)$ , where  $S^{(i,j)}(n)$  is the direction phase of the (i, j)th map with the same definition as that for the single logistic map. From Fig. 3, we can see that, in the weak coupling range  $\epsilon < 0.12$ , there always exists a sharp transition of M at a certain value of  $\mu$  for each coupling. However, the magnitude of M decreases as the coupling  $\epsilon$  increases. Because of the coupling, there are no periodic windows of the direction phase as found in the case of the single map. Besides, the critical value of  $\mu$  for the transition increases as the coupling increases (see inset in Fig. 3). All these clearly indicate that the net direction phase M characterizes the statistical behavior of the lattices. As long as the coupling is weak, the lattices basically show statistical behavior similar to that of the single map, and the symmetry breaking shifts to a large value of  $\mu$  as the coupling increases. In addition, the hierarchical feature of the transition points (in Fig. 2) disappears since the detailed bifurcations are destroyed by the mean-field effect of the coupling.

When  $\epsilon > 0.15$ , there is no net direction phase, i.e.,  $M \approx 0$ , for the lattices (see Fig. 3). However, due to the strong coupling, the lattices show synchronous direction phases, or all (or most) of the individual maps in the system have the same phases, up or down. In Fig. 4, we show



FIG. 4. The snapshots of the direction phases of Eq. (1) ( $\mu = 3.7$ ) with couplings: (a)  $\epsilon = 0$ ; (b)  $\epsilon = 0.3$ ; (c)  $\epsilon = 0.6$ . (d) The values of  $X_n(i, j)$  at the same time *n* as for (c). Different shades of gray represent different ranges of values of  $X_n(i, j)$ .



FIG. 5.  $\Theta$  versus  $\mu$  for two couplings  $\epsilon = 0.2$ , 0.4. Inset: (a),(b) the net direction phase M(n) changes with the time *n* for two  $\mu$  values; (c) the onset value of  $\mu_{\theta}$  versus  $\epsilon$ ; (d) the scaling of  $(\Theta - \Theta_0) \sim (\mu - \mu_{\theta})^{\beta}$  with  $\Theta_0 = 0.056$ .

the snapshots of the direction phases and the values of  $X_n(i, j)$  for the lattices. We can see that as the coupling  $\epsilon$  increases the direction phases become synchronized from their initial randomness. There are many clusters which have the same direction phase or a phase-locked state. The sizes of the clusters increase as the coupling  $\epsilon$  increases, and at  $\epsilon = 0.6$  all maps have the same direction phase [see Fig. 4(c)]. Note that, although all of the maps are synchronized in their phases or are phase-locked together, the values of the maps  $X_n(i, j)$  are still chaotic and have a very weak correlation between each other [see Fig. 4(d)].

In order to characterize the phase synchronization or the phase-locked state between the maps, let us define an in-phase ratio  $\Theta = T^{-1} \sum_{n=1}^{T} |M(n)|$  as a measure of the synchronization, where  $M(n) = L^{-2} \sum_{i,j} S^{(i,j)}(n)$  is the net direction phase of the lattices at time n. Clearly,  $\Theta$  describes the synchronization. When  $\Theta = 1$ , all maps have the same direction phase, i.e., an in-phase synchronization while, when  $\Theta = 0$ , half-maps have the same direction phases. Figure 5 shows two examples of  $\Theta$  versus  $\mu$ . We can see that, for a strong coupling  $\epsilon = 0.4$ . there is a strong in-phase synchronization with  $\Theta \sim 1$  when  $\mu < 3.82$ . All maps have a up phase at time n and then have a down phase at time n + 1 [see inset (a) and also Fig. 4(c)]. The lattices show a synchronous oscillation in their direction phases. However, there is a transition near  $\mu = 3.82$  from a strong in-phase synchronization to an antiphase one. In the antiphase state there are many clusters, and some clusters have an up phase and some have a down phase. Thus the net direction phase  $M(n) \sim 0$  or the in-phase ratio  $\Theta \sim 0$  [see the inset (b)]. Physically, this is because the strong coupling has an effect which forces the nearest neighboring maps to be in phase, while the maps with a large value of  $\mu$  behave as random phases. The insets (c) and (d) in Fig. 5 show the transition point

 $\mu_{\Theta}$  versus the coupling  $\epsilon$  and the scaling behavior near the transition  $\Theta \sim (\mu - \mu_{\Theta})^{\beta}$  with  $\beta = 1.96 \pm 0.03$ . Nevertheless, there is no such transition for a weak coupling with  $\epsilon < 0.3$ . The cluster sizes are small, and there is also some randomness in their direction phases for each cluster.

In conclusion, a transition of the direction phases from an ordered state to a disordered one for a logistic map is studied. The direction phases in a coupled map lattice are found to behave as a synchronous oscillation or a cluster ordering feature, which is relevant to the pattern formation of the dynamical system. However, the transition is not a phase transition since the increase of the value of Mappears in a situation where the order is reduced, i.e., a regular arrangement of the up and down direction phases becomes irregular. Finally, it is noted that the translation of the real-valued trajectory into the direction phases is a symbolic encoding. That is, by the definition of S(n), a binary partition with elements S(n) = 1 or -1 in the phase space is defined, and the trajectory is encoded according to which partition element a point is in. Such a symbolic encoding may be applicable for the dynamical systems and can be discussed using the information theory [13].

We thank B. L. Hao and B. W. Li for helpful discussions, and H. Kantz for useful comments and suggestions. This work was supported by the NNSF (No. 19625409), the Nonlinear Project of the NSTC, and the HKRGC.

- [1] M.J. Feigenbaum, J. Stat. Phys. 19, 25 (1978).
- [2] B.L. Hao, Elementary Symbolic Dynamics and Chaos in Dissipative Systems (World Scientific, Singapore, 1989).
- [3] E. Ott, *Chaos in Dynamical System* (Cambridge University Press, New York, 1993); H. A. Lauwerier, in *Chaos*, edited by A. V. Holden (Manchester University Press, Manchester, 1986).
- [4] S.K. Ma, Modern Theory of Critical Phenomena (Benjamin, London, 1976).
- [5] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. 65, 851 (1993).
- [6] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. 76, 1804 (1996).
- [7] It is easy to obtain  $X_f = (\mu 1)/\mu$ ,  $X_c = [1 (\mu^2 4\mu + 4)^{1/2}]/2$ ,  $X_{\text{max}} = \mu/4$ , and  $X_{\text{min}} = \mu^2(4 \mu)/16$ .
- [8] H. Kantz (private communication).
- [9] C. Grebogi *et al.*, Phys. Rev. Lett. **48**, 1507 (1982);
  C. Grebogi *et al.*, Phys. Rev. A **36**, 5365 (1987);
  H. Stewart *et al.*, Phys. Rev. Lett. **75**, 2478 (1995).
- [10] P. Grassberger and T. Schreiber, Physica (Amsterdam) 50D, 177 (1991); H. Chate and P. Manneville, Physica (Amsterdam) 32D, 409 (1988).
- [11] A. S. Pikovsky *et al.*, Phys. Rev. Lett. **79**, 47 (1997); M. G. Rosenblum *et al.*, Phys. Rev. Lett. **78**, 4193 (1997); E. Rosa *et al.*, Phys. Rev. Lett. **80**, 1642 (1998).
- [12] K. Kaneko, *Theory and Applications of Coupled Map Lat*tices (Wiley, New York, 1993); Chaos 2, 279 (1992).
- [13] C. Shannon and W. Weaver, *The Mathematical Theory* of Communication (University of Illinois Press, Urbana, 1949).