Chaotic Transport and Current Reversal in Deterministic Ratchets

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We address the problem of the classical deterministic dynamics of a particle in a periodic asymmetric potential of the ratchet type. We take into account the inertial term in order to understand the role of the chaotic dynamics in the transport properties. By a comparison between the bifurcation diagram and the current, we identify the origin of the current reversal as a bifurcation from a chaotic to a periodic regime. Close to this bifurcation, we observed trajectories revealing intermittent chaos and anomalous deterministic diffusion.

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In recent years there has been an increasing interest in the study of the transport properties of nonlinear systems that can extract usable work from unbiased nonequilibrium fluctuations. These so-called ratchet systems can be modeled, for instance, by considering a Brownian particle in a periodic asymmetric potential and acted upon by an external time-dependent force of zero average [1,2]. This recent burst of work is motivated in part by the challenge to explain the unidirectional transport of molecular motors in the biological realm [3]. Another source of motivation arises from the potential for new methods of separation or segregation of particles [4], and more recently in the recognition of the "ratchet effect" in the quantum domain [5]. The latter research includes a quantum ratchet based on an asymmetric (triangular) quantum dot [6], an asymmetric antidot array [7], the ratchet effect in surface electromigration [8], a ratchet potential for fluxons in Josephson-junctions arrays [9], a ratchet effect in cold atoms using an asymmetric optical lattice [10], and the reducing of vortex density in superconductors using the ratchet effect [11].

In order to understand the generation of unidirectional motion from nonequilibrium fluctuations, several models have been used. In Ref. [1], there is a classification of different types of ratchet systems; among them we can mention the "rocking ratchets," in which the particles move in an asymmetric periodic potential subject to spatially uniform, time-periodic deterministic forces of zero average. Most of the models, so far, deal with the overdamped case in which the inertial term due to the finite mass of the particle is neglected. However, in recent studies, this oversimplification was overcome by treating properly the effect of finite mass [5,12,13].

In particular, in a recent paper [12], Jung, Kissner, and Hänggi study the effect of finite inertia in a deterministically rocked, periodic ratchet potential. They consider the deterministic case in which noise is absent [14]. The inertial term allows the possibility of having both regular and chaotic dynamics, and this deterministically induced chaos can mimic the role of noise. They showed that the system can exhibit a current flow in either direction, presenting multiple current reversals as the amplitude of the external force is varied.

In this paper, the problem of transport in periodic asymmetric potentials of the ratchet type is address. We elaborate on the model analyzed by Jung *et al.* [12], in which they find multiple current reversals in the dynamics. In fact, the study of the current-reversal phenomena has given rise to a research activity on its own [1].

The goal of this paper is to reveal the origin of the current reversal, by analyzing in detail the dynamics for values of the parameters just before and after the critical values at which the current reversal takes place.

Let us consider the one-dimensional problem of a particle driven by a periodic time-dependent external force, under the influence of an asymmetric periodic potential of the ratchet type. The time average of the external force is zero. Here, we do not take into account any kind of noise, and thus the dynamics is deterministic. The equation of motion is given by

$$m\ddot{x} + \gamma \dot{x} + \frac{dV(x)}{dx} = F_0 \cos(\omega_D t), \qquad (1)$$

where m is the mass of the particle, γ is the friction coefficient, $V(x)$ is the external asymmetric periodic potential, F_0 is the amplitude of the external force, and ω_D is the frequency of the external driving force. The ratchet potential is given by

$$V(x) = V_1 - V_0 \sin \frac{2\pi(x - x_0)}{L} - \frac{V_0}{4} \sin \frac{4\pi(x - x_0)}{L}, \qquad (2)$$

where L is the periodicity of the potential, V_0 is the amplitude, and V_1 is an arbitrary constant. The potential is shifted by an amount x_0 in order that the minimum of the potential is located at the origin.

Let us define the following dimensionless units: $x' = x/L$, $x_0' = x_0/L$, $t' = \omega_0 t$, $w = \omega_D/\omega_0$, $b = \gamma/m\omega_0$, and $a = F_0/mL\omega_0^2$. Here, the frequency ω_0 is given by $\omega_0^2 = 4\pi^2 V_0 \delta/mL^2$ and δ is defined by $\delta = \sin(2\pi|x_0'|) + \sin(4\pi|x_0'|)$.

The frequency ω_0 is the frequency of the linear motion around the minima of the potential; thus we are scaling

the time with the natural period of motion $\tau_0 = 2\pi/\omega_0$. The dimensionless equation of motion, after renaming the variables again without the primes, becomes

$$\ddot{x} + b\dot{x} + \frac{dV(x)}{dx} = a\cos(wt), \qquad (3)$$

where the dimensionless potential [15] is given by $V(x) = C - [\sin 2\pi(x - x_0) + 0.25\sin 4\pi(x - x_0)]/4\pi^2\delta$ and is depicted in Fig. 1.

In the equation of motion, Eq. (3), there are three dimensionless parameters: a, b, and w, defined above in terms of physical quantities. We vary the parameter a and fix $b = 0.1$ and $w = 0.67$ throughout this paper.

The extended phase space in which the dynamics is taking place is three dimensional, since we are dealing with an inhomogeneous differential equation with an explicit time dependence. This equation can be written as a three-dimensional dynamical system that we solve numerically, using the fourth-order Runge-Kutta algorithm. The equation of motion Eq. (3) is nonlinear and thus allows the possibility of chaotic orbits. If the inertial term associated with the second derivative $\ddot{x}$ were absent, then the dynamical system could not be chaotic.

The main motivation behind this work is to study in detail the origin of the current reversal in a chaotically deterministic rocked ratchet. In order to do so, we have to study first the current J itself, which we define as the time average of the average velocity over an ensemble of initial conditions. Therefore, the current involves two different averages: the first average is over M initial conditions, which we take equally distributed in space, centered around the origin, and with an initial velocity equal to zero. For a fixed time, say t_j, we obtain an average velocity that we denoted as v_j, and is given by $v_j = \frac{1}{M}\sum_{i=1}^{M}\dot{x}_i(t_j)$. The second average is a time average; since we take a discrete time for the numerical solution of the equation of motion, we have a discrete finite set of N different times t_j, and then the current can be defined as $J = \frac{1}{N}\sum_{j=1}^{N}v_j$. This quantity is a single number for a fixed set of parameters a, b, w, but it varies with the parameter a, fixing b and w.
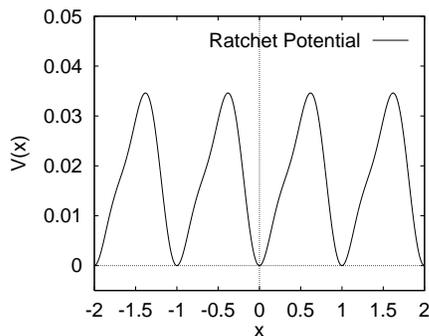
Besides the continuum orbits in the extended phase space, we can obtain the Poincaré section, using as a stroboscopic time the period of oscillation of the external force. With the aid of Poincaré sections we can distinguish between periodic and chaotic orbits, and we can obtain a bifurcation diagram as a function of the parameter a.

The bifurcation diagram for $b = 0.1$ and $w = 0.67$ is shown in Fig. 2a in a limited range of the parameter a. We can observe a period-doubling route to chaos, and after a chaotic region, there is a bifurcation taking place at a critical value $a_c \simeq 0.08092844$. It is precisely at this bifurcation point that the current reversal occurs. After this bifurcation, a periodic window emerges, with an orbit of period four. In Fig. 2b, we show the current as a function of the parameter a, in exactly the same range as the bifurcation diagram above. We notice the abrupt transition at the bifurcation point that leads to the first current reversal. In Figs. 2a and 2b we are analyzing only a short range of values of a, where the first current reversal takes place. If we vary a further, we can obtain multiple current reversals [12].

In order to understand in more detail the nature of the current reversal, let us look at the orbits just before and after the transition. The reversal occurs at the critical value $a_c \simeq 0.08092844$. If a is below this critical value a_c, say $a = 0.074$, then the orbit is periodic, with period two. For this case we depict, in Fig. 3a, the position of the particle as a function of time. We notice a period-two orbit, as can be distinguished in the bifurcation diagram for
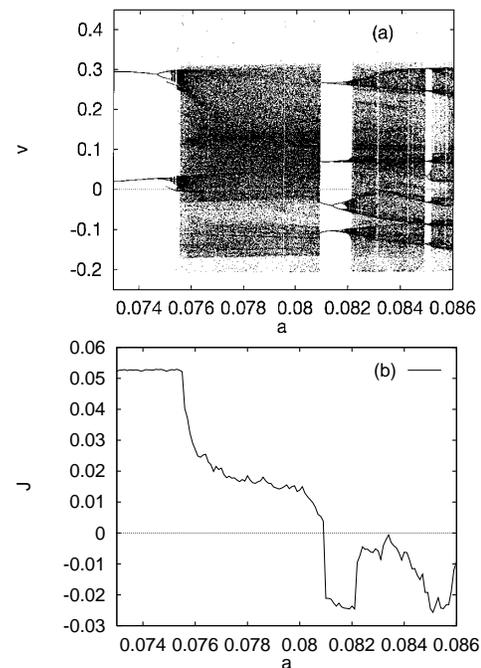

FIG. 1. The dimensionless ratchet periodic potential $V(x)$.

FIG. 2. For $b = 0.1$ and $w = 0.67$ we show the following: (a) The bifurcation diagram as a function of a, and (b) the current J as a function of a. The range in the parameter a corresponds to the first current reversal.
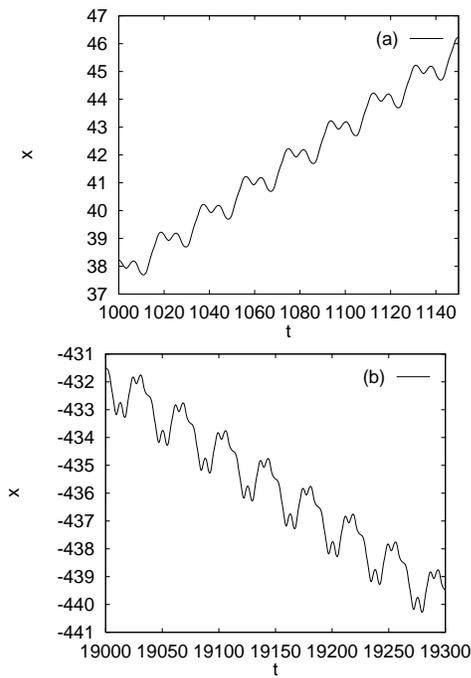
FIG. 3. For $b = 0.1$ and $w = 0.67$ we show the following: (a) The trajectory of the particle for $a = 0.074$ (positive current), and (b) the trajectory for $a = 0.081$ (negative current).

$a = 0.074$. In Fig. 3b we show again the position as a function of time for $a = 0.081$, which is just above the critical value a_c. In this case, we observe a period-four orbit that corresponds to the periodic window in the bifurcation diagram in Fig. 2a. This orbit is such that the particle is "climbing" in the negative direction, that is, in the direction in which the slope of the potential is higher. We notice that there is a qualitative difference between the periodic orbit that transports particles to the positive direction and the periodic orbit that transports particles to the negative direction: in the latter case, the particle requires twice the time than in the former case, to advance one well in the ratchet potential. A closer look at the trajectory in Fig. 3b reveals the "trick" that the particle uses to navigate in the negative direction: in order to advance one step to the left, it moves first one step to the right and then two steps to the left. The net result is a negative current.

In Fig. 4, we show a typical trajectory for a just below a_c. The trajectory is chaotic and the corresponding chaotic attractor is depicted in Fig. 5. In this case, the particle starts at the origin with no velocity; it jumps from one well in the ratchet potential to another well to the right or to the left in a chaotic way. The particle gets trapped oscillating for a while in a minimum (sticking mode), as is indicated by the integer values of x in the ordinate, and suddenly starts a running mode with average constant velocity in the negative direction. In terms of the velocity, these running modes, as the one depicted in Fig. 3b, correspond to periodic motion. The phenomenology can be described as follows. For values of a above a_c, as in
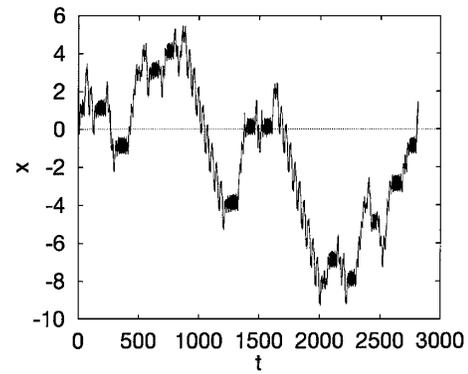

FIG. 4. The intermittent chaotic trajectory of the particle for $b = 0.1$, $w = 0.67$, and $a = 0.08092844$.

Fig. 3b, the attractor is a periodic orbit. For a slightly less than a_c there are long stretches of time (running or laminar modes) during which the orbit appears to be periodic and closely resembles the orbit for $a > a_c$, but this regular (approximately periodic) behavior is intermittently interrupted by finite duration "bursts" in which the orbit behaves in a chaotic manner. The net result in the velocity is a set of periodic stretches of time interrupted by a burst of chaotic motion, signaling precisely the phenomenon of intermittency [16]. As a approaches a_c from below, the duration of the running modes in the negative direction increases, until the duration diverges at $a = a_c$, where the trajectory becomes truly periodic.

To complete this picture, in Fig. 5, we show two attractors: (1) the chaotic attractor for $a = 0.08092$, just below a_c, corresponding to the trajectory in Fig. 4, and (2) the period-four attractor for $a = 0.08093$, corresponding to the trajectory in Fig. 3b. This periodic attractor consists of four points in phase space, which are located at the center of the open circles. We obtain these attractors confining the dynamics in x between -0.5 and 0.5. As a approaches a_c from below, the dynamics in the attractor becomes intermittent, spending
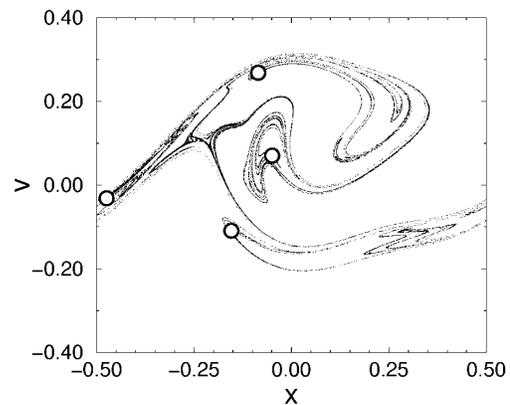

FIG. 5. For $b = 0.1$ and $w = 0.67$ we show two attractors: a chaotic attractor for $a = 0.08092$, just below a_c, and a period-four attractor consisting of four points located at the center of the open circles.

most of the time in the vicinity of the period-four attractor, and suddenly "jumping" in a chaotic way for some time, and then returning close to the period-four attractor again, and so on. In terms of the velocity, the result is an intermittent time series as discussed above.

In order to characterize the deterministic diffusion in this regime, we calculate the mean square displacement $\langle x^2 \rangle$ as a function of time. We obtain numerically that $\langle x^2 \rangle \sim t^\alpha$, where the exponent $\alpha \simeq 3/2$. This is a signature of anomalous deterministic diffusion, in which $\langle x^2 \rangle$ grows faster than linear, that is, $\alpha > 1$ (superdiffusion). Normal deterministic diffusion corresponds to $\alpha = 1$. In contrast, the trajectories in Figs. 3a and 3b transport particles in a ballistic way, with $\alpha = 2$. The relationship between anomalous deterministic diffusion and intermittent chaos has been explored recently, together with the connection with Lévy flights [17]. The character of the trajectories, as the one in Fig. 4, remains to be analyzed more carefully in order to determine if they correspond to Lévy flights.

In summary, we have identified the mechanism by which the current reversal in deterministic ratchets arises: it corresponds to a bifurcation from a chaotic to a periodic regime. Near this bifurcation, the chaotic trajectories exhibit intermittent dynamics and the transport arises through deterministic anomalous diffusion with an exponent greater than one (superdiffusion). As the control parameter a approaches the critical value a_c at the bifurcation from below, the duration of the running modes in the negative direction increases. Finally, the duration diverges at the critical value, leading to a truly periodic orbit in the negative direction. This is precisely the mechanism by which the current reversal takes place.

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