Classical Teleportation of a Quantum Bit

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Classical teleportation is defined as a scenario where the sender is given the classical description of an arbitrary quantum state while the receiver simulates any measurement on it. This scenario is shown to be achievable by transmitting only a few classical bits if the sender and receiver initially share local hidden variables. Specifically, a communication of 2.19 bits is sufficient on average for the classical teleportation of a qubit, when restricted to von Neumann measurements. The generalization to positive-operator-valued measurements is also discussed.

The past few years have seen the emergence of quantum information theory. This generalization of Shannon's classical theory of information describes how quantum systems can be manipulated from an information-theoretic perspective. In particular, whereas the basic quantity of Shannon's theory is the amount of information that can be stored in a two-state register (a bit), the basic quantity of quantum information theory is the amount of information that is contained in a quantum system whose Hilbert space is two dimensional (called a qubit). Also, quantum information theory deals with completely novel properties that have no classical counterpart such as quantum entanglement (nonlocal correlations), and their role in communication problems.

The exploration of the relation between classical and quantum informational resources is a central objective of this new discipline. Perhaps one of the most celebrated examples of such a relation is *quantum teleportation* [1], in which the physical transmission of a qubit is replaced by the transmission of two classical bits supplemented with preexisting entanglement. The analysis of this process raises a fundamental question: What is the exact information content carried by a qubit? On the one hand, specifying a pure state in two-dimensional Hilbert space, i.e., $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$, requires two real numbers (the angles θ and ϕ in the Bloch sphere), so that communicating a *known* quantum state apparently requires an *infinite* number of classical bits. On the other hand, it happens that teleportation can be achieved by transmitting only two classical bits, provided that they are supplemented with a previously existing entangled pair of qubits. Yet, as such, an entangled pair cannot do more than generate correlated random numbers, and surely not be used to communicate—this would violate causality. This state of affairs led Vaidman to question "whether the essence of a quantum state is only two bits" [2].

In this Letter, we shall show that the situation is even more surprising than envisaged by Vaidman. We shall exhibit a scenario, denoted as *classical teleportation,* in which a quantum entangled pair is not even necessary, and

the teleportation of a *known* qubit can be simulated entirely classically. The sender, called Alice, is given the classical description of a quantum state, i.e., a vector $\vec{a}$ on the Bloch sphere. The receiver, Bob, chooses a particular measurement, that is, a vector $\vec{b}$ on the Bloch sphere in the case of a von Neumann measurement. Importantly, Bob's choice is unknown to Alice, while Bob is ignorant of Alice's quantum state. Bob's task is to give an outcome that is consistent with him having performed a measurement on Alice's state. Here, we will show that if Alice and Bob share initially some local "hidden" variables (LHV), that is, they possess an identical (possibly infinite) list of random numbers, then carrying out classical teleportation requires only a finite—and very limited—amount of communication. In other words, all possible measurements performed by Bob can be simulated by Alice transmitting a few classical bits to Bob, *as if* the qubit had actually been teleported. We find that, if Bob restricts himself to von Neumann measurements, an average of 2.19 bits of classical communication is sufficient to complete this task, which is only 0.19 bits more than the amount needed for standard quantum teleportation [1]. More generally, if Bob's measurement is based on a positive-operator-valued measure (POVM), the best algorithm we have found is less efficient and uses a total of 6.38 bits of two-way communication.

Note, however, that if Alice and Bob *do* share prior entanglement, the task in which Bob must simulate the statistics of any measurement on a quantum bit *known* by Alice (but unknown to him) can be achieved with only one bit of classical communication. Indeed, it is enough for Alice to measure her entangled qubit in an appropriate basis (made of the known state and its orthogonal complement) and to communicate the outcome to Bob [3]. From this point of view, the additional cost associated with the use of shared LHVs instead of entanglement is more properly counted as 1.19 bits. Furthermore, it should be emphasized that our scheme cannot simulate all aspects of the teleportation of a qubit, e.g., entanglement swapping [4]. Nevertheless, it is remarkable that achieving classical teleportation using one-way classical communication *but* with no shared

LHVs requires in contrast the transmission of an *infinite* number of bits (Alice should then transmit to Bob the description of her state, i.e., the angles θ and ϕ).

The protocol we shall present below is inspired by, and closely related to, schemes for classically simulating Bell correlations recently discovered by Brassard *et al.* [5] and independently by Steiner [6]. It is well known that any local-hidden-variable model of quantum mechanics *cannot* reproduce quantum correlations, as reflected by the violation of Bell inequalities. Surprisingly, these authors have demonstrated that supplementing a local-hidden-variable model with a very limited amount of classical communication suffices to perfectly simulate quantum correlations. Our protocol for the classical teleportation of a qubit is based on the generalization of Steiner's protocol to the Bloch sphere (Steiner's protocol was restricted to *real* quantum mechanics, i.e., $\phi = 0$) presented in Ref. [7], where it was used to analyze the detection loophole in Bell experiments. It differs from that scheme by the use of *triplets* of orthogonal vectors so as to span the Bloch sphere more uniformly. It also decreases significantly the amount of communication needed by better exploiting the shared variables between Alice and Bob.

Before presenting the details of our classical scenario, let us highlight its central features. Initially, Alice and Bob share an infinite list of triplets of normalized vectors distributed randomly on the Bloch sphere, $\{\vec{\lambda}_k, \vec{\mu}_k, \vec{\nu}_k\}$, with $k = 1, 2 \ldots$. For each k, the three vectors form an orthogonal frame oriented randomly and chosen independently of the other triplets. In addition, Alice and Bob also possess a common infinite list of independent random numbers u_k (with $k = 1, 2, \ldots$) uniformly distributed in the interval $[0, \sqrt{3}]$. Together the infinite set of $(\vec{\lambda}_k, \vec{\mu}_k, \vec{\nu}_k, u_k)$ constitutes the LHVs [8]. Once Alice and Bob are separated, Alice is given an arbitrary normalized vector $\vec{a}$ on the Bloch sphere specifying the quantum state $|\psi(\vec{a})\rangle$ to be communicated to Bob, while Bob chooses an arbitrary von Neumann measurement (described by vector $\vec{b}$). Each party is oblivious of the choice the other party has made. Thus, the quantum state chosen by Alice (and unknown to Bob) is characterized by the projector $P_{\vec{a}} = (\mathbb{1} + \vec{a} \cdot \vec{\sigma})/2$, where $\vec{\sigma}$ denotes the Pauli matrices. The measurement chosen by Bob (unknown to Alice) corresponds to the observable $\sigma_{\vec{b}} = \vec{b} \cdot \vec{\sigma}$, resulting in outcome $\pm$ with probability

$$p(\pm \mid \vec{a}) = \mathrm{Tr}(P_{\pm \vec{b}} P_{\vec{a}}) = \frac{1 \pm \vec{a} \cdot \vec{b}}{2}, \qquad (1)$$

where $P_{\pm \vec{b}} = (\mathbb{1} \pm \vec{b} \cdot \vec{\sigma})/2$. The goal is for Bob to generate a measurement statistics *identical* to that predicted by quantum mechanics, Eq. (1), but without Alice actually transmitting her qubit to Bob. Instead, the variables $(\vec{\lambda}_k, \vec{\mu}_k, \vec{\nu}_k, u_k)$ are supplemented with classical communication. In addition, we will assume that the scenario is repeated many times, that is, Alice prepares N qubits

in states $|\psi(\vec{a}_1)\rangle, |\psi(\vec{a}_2)\rangle, \ldots, |\psi(\vec{a}_N)\rangle$, and sends them to Bob, whereupon Bob carries out separate measurements $\sigma_{\vec{b}_1}, \sigma_{\vec{b}_2}, \ldots, \sigma_{\vec{b}_N}$ on each qubit. The transmission of N qubits is simulated in parallel, allowing the protocol to make use of block coding, hence minimizing the amount of classical communication needed.

Our protocol works as follows. First, Alice divides the interval $[0, \sqrt{3}]$ of each u_k into four segments. These segments depend on the orientation of the corresponding variables $(\vec{\lambda}_k, \vec{\mu}_k, \vec{\nu}_k)$ with respect to $\vec{a}$ as

zone A_λ: $0 \leq u_k < |\vec{a} \cdot \vec{\lambda}_k|$,

zone A_μ: $0 \leq u_k - |\vec{a} \cdot \vec{\lambda}_k| < |\vec{a} \cdot \vec{\mu}_k|$, $\qquad (2)$

zone A_ν: $0 \leq u_k - |\vec{a} \cdot \vec{\lambda}_k| - |\vec{a} \cdot \vec{\mu}_k| < |\vec{a} \cdot \vec{\nu}_k|$,

zone R: $|\vec{a} \cdot \vec{\lambda}_k| + |\vec{a} \cdot \vec{\mu}_k| + |\vec{a} \cdot \vec{\nu}_k| \leq u_k < \sqrt{3}$.

Alice then performs the following operations:

$A1$: She sets the index $k = 1$.

$A2$: She checks whether u_k belongs to zone R. If it does, she *rejects* the triplet $(\vec{\lambda}_k, \vec{\mu}_k, \vec{\nu}_k)$, that is, she increments k by one and goes back to step $A2$.

$A3$: Once the kth triplet has been *accepted,* Alice communicates to Bob at which iteration u_k first belonged to one of the zones A_λ, A_μ, or A_ν, and to which of these zones it belongs.

$A4$: She also sends an extra bit which is the sign of $\vec{a} \cdot \vec{\lambda}_k$ if $u_k \in A_\lambda$, or similarly for A_μ and A_ν.

Upon receiving this information from Alice, Bob knows which vector $\vec{\lambda}_k$, $\vec{\mu}_k$, or $\vec{\nu}_k$ was accepted by Alice. Denote this vector by $\vec{\lambda}$. He then carries out the following operations:

$B1$: He flips $\vec{\lambda} \rightarrow -\vec{\lambda}$ if the sign of $\vec{a} \cdot \vec{\lambda}$ is negative.

$B2$: The outcome $\pm$ of Bob's measurement of $\sigma_{\vec{b}}$ is then given by the sign of $\vec{b} \cdot \vec{\lambda}$.

Let us check that, with this protocol, Bob correctly reproduces the quantum probabilities. Since the random variable u_k is uniformly distributed, the probability of accepting $\vec{\lambda}_k$ conditionally on $\vec{a}$, i.e., $p(\vec{\lambda}_k \mid \vec{a}) = |\vec{a} \cdot \vec{\lambda}_k|/(4\pi\sqrt{3})$. Using the isotropy of the $\vec{\lambda}_k$ distribution, the average probability of acceptance (in zone A_λ) is given by

$$p_{A_\lambda} = \int d\vec{\lambda} \, p(\vec{\lambda} \mid \vec{a}) = \frac{1}{2\sqrt{3}} \qquad (3)$$

at each iteration k. Note that it does not depend on $\vec{a}$. Similarly, we have $p(\vec{\mu}_k \mid \vec{a}) = |\vec{a} \cdot \vec{\mu}_k|/(4\pi\sqrt{3})$ and $p(\vec{\nu}_k \mid \vec{a}) = |\vec{a} \cdot \vec{\nu}_k|/(4\pi\sqrt{3})$, so that $p_{A_\mu} = p_{A_\nu} = 1/(2\sqrt{3})$. Thus, the average probability of acceptance (in any zone) is $p_A = p_{A_\lambda} + p_{A_\mu} + p_{A_\nu} = \sqrt{3}/2$, while the corresponding average probability of rejection is simply $p_R = 1 - p_A = (2 - \sqrt{3})/2$. As far as the statistics of Bob's outcome is concerned, it is sufficient to consider one of the vectors $\vec{\lambda}_k$, $\vec{\mu}_k$, and $\vec{\nu}_k$, since the correlation of $\vec{\lambda}$ with $\vec{a}$ expressed by $p(\vec{\lambda} \mid \vec{a}) = |\vec{a} \cdot \vec{\lambda}|/(4\pi\sqrt{3})$ is independent of whether $\vec{\lambda} = \vec{\lambda}_k$, $\vec{\mu}_k$, or $\vec{\nu}_k$. Once $\vec{\lambda}$ has

been accepted, its *a posteriori* probability distribution is given by

$$P(\text{accepted } \vec{\lambda} \,|\, \vec{a}) = \frac{p(\vec{\lambda} \,|\, \vec{a})}{p_{A_\lambda}} = \frac{|\vec{a} \cdot \vec{\lambda}|}{2\pi}. \quad (4)$$

Consequently, the probability that Bob chooses outcome $\pm$ for a given $\vec{a}$ is

$$p(\pm \,|\, \vec{a}) = \int d\vec{\lambda} \, P(\text{accepted } \vec{\lambda} \,|\, \vec{a})$$
$$\times \, [\Theta(\vec{a} \cdot \vec{\lambda})\Theta(\pm\vec{b} \cdot \vec{\lambda}) $$
$$+ \, \Theta(-\vec{a} \cdot \vec{\lambda})\Theta(\mp\vec{b} \cdot \vec{\lambda})], \quad (5)$$

where $\Theta(\cdot)$ is the Heaviside step function. To verify that this integral coincides with the predictions of quantum mechanics, Eq. (1), it is easiest to calculate $p(+ \,|\, \vec{a}) - p(- \,|\, \vec{a}) - p(+ \,|\, -\vec{a}) + p(- \,|\, -\vec{a})$ and use the symmetry of the problem (see [7]).

We now compute the amount of classical information that must be communicated by Alice to Bob in order to carry out this protocol. It will appear that the use of three vectors $\vec{\lambda}_k$, $\vec{\mu}_k$, and $\vec{\nu}_k$ is helpful to reduce this amount. Ignoring the extra bit (the sign of $\vec{a} \cdot \vec{\lambda}$) for the moment, the information needed by Bob to choose an outcome is the value of k (the iteration at which a triplet has been accepted) and whether $\vec{\lambda}_k$, $\vec{\mu}_k$, or $\vec{\nu}_k$ was accepted. Denote the latter variable as $l = 1, 2, 3$. If block coding is used, the minimum number of bits that must be communicated per simulation of one qubit is given asymptotically by the Shannon entropy of k and l. A crucial point is that the random number u_k used by Alice to accept or reject the kth triplet is also known to Bob, so that the latter can infer the *a priori* probabilities of Alice choosing $\vec{\lambda}_k$, $\vec{\mu}_k$, or $\vec{\nu}_k$. Therefore, the *minimum* amount of information that must be communicated on average is the entropy of k and l *conditional* on the infinite set of variables u_k's, that is, the uncertainty on (k, l) when $u_1, u_2, \ldots$ are known.

To calculate this quantity, note that, for a given set $u_1, u_2, \ldots$, the probability of accepting A_l at the kth iteration is given by

$$p(k, l \,|\, u_1, u_2, \ldots) = p(R \,|\, u_1) \cdots p(R \,|\, u_{k-1})p(A_l \,|\, u_k), \quad (6)$$

where $p(A_l \,|\, u)$ is the *a priori* probability of accepting A_l knowing u, and $p(R \,|\, u)$ is the *a priori* probability of rejection for a given u. For instance, as $\vec{\lambda}$ is uniformly distributed on the Bloch sphere, we have

$$p(A_\lambda \,|\, u) = \int \frac{d\vec{\lambda}}{4\pi} \Theta(|\vec{a} \cdot \vec{\lambda}| - u). \quad (7)$$

Similar expressions can be obtained from Eq. (2) for $p(A_\mu \,|\, u)$, $p(A_\nu \,|\, u)$, and $p(R \,|\, u)$. Since these four probabilities are not equal, block coding allows Alice to compress the information sent to Bob. The entropy of k and l conditional on $u_1, u_2, \ldots$ (averaged over these variables) is given by

$$H \equiv H(k, l \,|\, u_1, u_2, \ldots) = -\int_0^{\sqrt{3}} \frac{du_1}{\sqrt{3}} \int_0^{\sqrt{3}} \frac{du_2}{\sqrt{3}} \cdots$$
$$\times \sum_{k=1}^{\infty} \sum_{l=1}^{3} p(k, l \,|\, u_1, u_2, \ldots) \log_2 p(k, l \,|\, u_1, u_2, \ldots). \quad (8)$$

Using Eq. (6), we get

$$H = \sum_{k=1}^{\infty} \sum_{l=1}^{3} [(k - 1)p_R^{k-2} p_{A_l} q_R + p_R^{k-1} q_{A_l}], \quad (9)$$

where

$$p_{A_l} = \int_0^{\sqrt{3}} \frac{du}{\sqrt{3}} \, p_{A_l}(u),$$
$$q_{A_l} = -\int_0^{\sqrt{3}} \frac{du}{\sqrt{3}} \, p_{A_l}(u) \log_2 p_{A_l}(u) \quad (10)$$

and similarly for p_R and q_R. Carrying out the sum in Eq. (9), we obtain the entropy $H = (q_A + q_R)/p_A$, with $p_A = \sum_{l=1}^{3} p_{A_l} = \sqrt{3}/2$ and $q_A = \sum_{l=1}^{3} q_{A_l}$. The expressions for q_{A_l} and q_R can be estimated numerically, giving $q_{A_\lambda} = 0.207$ bits, $q_{A_\mu} = 0.366$ bits, $q_{A_\nu} = 0.341$ bits, and $q_R = 0.177$ bits. This results in $H = 1.19$ bits for the amount of information needed for Bob to know which $\vec{\lambda}$ was accepted. Consequently, taking into account the extra bit needed to communicate the sign of $\vec{a} \cdot \vec{\lambda}$, the total amount of classical information that must be transmitted is $1.19 + 1 = 2.19$ bits, as announced.

This scheme can also be applied to the simulation of Bell correlations in a singlet state, as in Refs. [5–7]. Suppose Alice and Bob have a classical protocol for simulating measurements on a singlet. If Alice measures $\sigma_{\vec{a}}$ and sends Bob an additional bit telling him which was the outcome of her measurement, then this is equivalent to Alice simulating the teleportation of a known qubit to Bob. Indeed, if Bob flips the outcome of his measurement conditionally on Alice's outcome, it is as if he had received the state $|\psi(\vec{a})\rangle$ as a consequence of the exact anticorrelations exhibited by a singlet. Accordingly, our classical teleportation protocol is equivalent to Alice simulating the measurement of $\sigma_{\vec{a}}$ on a singlet, supplemented by the transmission of one extra bit identifying which outcome she obtained. Hence, the simulation of the singlet with our scheme necessitates only 1.19 bits on average. For comparison, the protocol of Brassard *et al.* [5] uses exactly eight bits, Steiner's [6] uses 2.97 bits on average (when adapted to general von Neumann measurements), whereas the protocol of Ref. [7] uses two bits on average. Also, Cleve [9] proposed a protocol requiring slightly less than two bits on average.

The most general measurement that Bob can carry out on a qubit is based on a POVM. We will now show that such a measurement can also be simulated, at the cost of a larger amount of (two-way) communication. The elements B_j of Bob's POVM (with $\sum_j B_j = \mathbb{1}$) are taken

to be proportional to one-dimensional projectors (other POVM's can trivially be obtained from such maximally refined POVM's). They can be expressed in terms of vectors $\vec{b}_j$ in the Bloch sphere as $B_j = (|b_j|\mathbb{1} + \vec{b}_j \cdot \vec{\sigma})/2$, with the completeness conditions $\sum_j |b_j| = 2$ and $\sum_j \vec{b}_j = 0$. The protocol starts by Alice carrying out her part as in steps $A1$ to $A4$ above. We still assume that many qubits are teleported in parallel so that block coding can be used. Bob then performs the following operations:

$B1'$: Same as $B1$.

$B2'$: He chooses randomly the jth outcome of his POVM with probability $|b_j|/2$.

$B3'$: He checks whether outcome j is consistent with the accepted vector $\vec{\lambda}$ that was chosen by Alice. To this end, he computes the scalar product of $\vec{\lambda}$ with $\vec{b}_j$. If $\vec{\lambda} \cdot \vec{b} \geq 0$, he accepts outcome j and sends a bit 0 back to Alice to let her know that he has chosen an outcome.

$B4'$: If $\vec{\lambda} \cdot \vec{b}_j < 0$, he then sends a bit 1 to Alice to inform her that he was unable to choose an outcome.

If Alice receives a bit set to 0, she terminates. But if she receives a bit set to 1, she increments k by 1 and returns to step $A2$. Alice and Bob iterate this two-way protocol until Bob can choose an outcome.

Thus, the probability that Bob accepts λ and gives outcome j is

$$P(j \,|\, \vec{a}) = \frac{|b_j|}{2} \int d\vec{\lambda} \, P(\text{accepted } \vec{\lambda} \,|\, \vec{a})[\Theta(\vec{a} \cdot \vec{\lambda})\Theta(\vec{b}_j \cdot \vec{\lambda}) + \Theta(-\vec{a} \cdot \vec{\lambda})\Theta(-\vec{b}_j \cdot \vec{\lambda})] = \frac{|b_j| + \vec{a} \cdot \vec{b}_j}{4}. \quad (11)$$

Hence, the probability that Bob terminates at step $B3'$ (averaged over j) is $\sum_j P(j \,|\, \vec{a}) = 1/2$. The probability that Bob gives outcome j is therefore simply obtained by normalizing Eq. (11), giving $(|b_j| + \vec{a} \cdot \vec{b}_j)/2$, in accordance with the predictions of quantum mechanics. Finally, since the probability of termination is $1/2$, the average number of iterations needed before acceptance by Bob is two. The average number of bits transmitted is therefore twice the number of bits exchanged during one iteration, the latter consisting of 2.19 bits sent by Alice to Bob plus one bit sent by Bob to Alice. Thus, we obtain a total of $2(2.19 + 1) = 6.38$ bits.

In summary, we have exhibited a classical protocol based on a LHV model of quantum mechanics, which simulates the teleportation of an arbitrary (but known) qubit by use of classical communication. It requires only 2.19 bits of communication if Bob must simulate any von Neumann measurement (or 6.38 bits if Bob must simulate any POVM measurement). This reflects that local-hidden-variable models are somewhat "close" to quantum mechanics: a small amount of classical communication is sufficient to fill the gap between them, at least for low-dimensional systems. As the dimensionality increases, however, the amount of communication increases exponentially (see [5]). It should be emphasized that our protocol is most probably not optimal, so that the minimum amount of classical communication needed for the classical teleportation of a qubit may even be less than two bits. Still, it is surprisingly efficient: if we restrict to two the average number of communicated bits per simulation, the teleportation fidelity attained with our scheme equals $(2 + 0.19 \times 0.5)/2.19 = 0.957$ (the fidelity 0.5 is achievable with LHVs only).

Another issue raised by our study concerns the interpretation of the recent experimental realizations of teleportation [10]. Notwithstanding the fact that these experiments are remarkable verifications of quantum mechanics, it appears that one should be cautious when asserting that such realizations demonstrate entanglement in a way that *cannot* be simulated by classical means. If the teleported state is *unknown* to Alice, then there are upper bounds on the teleportation fidelity that can be obtained if only classical communication is allowed and no entanglement is used. In contrast, we have shown that if the state is *known* to Alice, the transmission of the state can be simulated classically using only a few bits of communication if the two parties share local hidden variables. This remark will hopefully shed some new light on the interpretation of these experiments.

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