# Limits for Entanglement Measures

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The basic principle of entanglement processing says that entanglement cannot increase under local operations and classical communication. Based on this principle, we show that *any* entanglement measure $E$ suitable for the regime of a high number of identically prepared entangled pairs satisfies $E_D \leq E \leq E_F$, where $E_D$ and $E_F$ are the entanglement of distillation and formation, respectively. Moreover, we exhibit a theorem establishing a very general form of bounds for distillable entanglement.

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Since the pioneering papers [1–3] on quantifying entanglement, much has been done in this field [4–12]. However, in the case of mixed states, we are still at the stage of gathering phenomenology. In the very fruitful axiomatic approach [4–6] there is not even an agreement as to what postulates should be satisfied by candidates for entanglement measures. Moreover, we do not know the quantum communication meaning of the known measures apart from entanglement of formation $E_F$ and entanglement of distillation $E_D$ [2], having the following dual meaning: (i) $E_D(\varrho)$ is the maximal number of singlets that can be produced from the state $\varrho$ by means of local operations and classical communication (LQCC). (ii) $E_F(\varrho)$ is the minimal number of singlets needed to produce the state $\varrho$ by LQCC operations. [More precisely, $E_D$ ($E_F$) is the minimal number of singlets *per copy* in the state $\varrho$ in the asymptotic sense of considering $n \to \infty$ copies altogether.] Unfortunately, they are very hard to deal with. One can ask a general question. Is there a rule that would somehow order the many possible measures satisfying some reasonable axioms? Moreover, is there any connection between the axiomatically defined measures and the entanglement of distillation and formation?

Surprisingly, it appears that just the two, historically first, measures of entanglement [2] constitute the sought after rule, being *extreme* measures. In this paper we show that any measure satisfying certain natural axioms (two of them specific to the *asymptotic regime* of a high number of identically prepared entangled pairs) must be confined between $E_D$ and $E_F$:

$$E_D \leq E \leq E_F . \qquad (1)$$

The result is compatible with some earlier results in this direction. In Ref. [13] Plenio and Vedral provided heuristic argumentation that an additive measure of entanglement should be no less than $E_D$. Uhlmann showed that nonregularized entanglement of formation [2] (closely related to $E_F$) is the upper bound for all convex functions which agree with it on pure states [14]. Finally, the presented result is compatible with the result by Popescu and Rohrlich [4], completed by Vidal [9], stating the uniqueness of the entanglement measure for pure states.

The proof of the result (contained in Theorem 1) is very simple, but it is very powerful. Indeed, as a by-product, we obtain (Theorem 2) surprisingly weak conditions for a function to be the upper bound for $E_D$. This is a remarkable result, as the evaluation of $E_D$ is one of the central tasks of the present stage of quantum entanglement theory. In particular, we obtain elementary proof that the relative entropy entanglement $E_r$ [5,6] and the function considered by Rains [11] are bounds for distillable entanglement. Note that the proof of Ref. [11] involves complicated mathematics, while the one of Ref. [6] is based on still unproven additivity assumption. In addition, our result is very general, and we expect it will result in an easy search for bounds on distillable entanglement. It is crucial that the basic tool we employ to obtain the results is the fundamental principle of entanglement theory stating that *entanglement cannot increase under local operations and classical communication* [1,2,4]. Thus the principle, putting bounds for the efficiency of distillation, plays a similar role to that of the second law of thermodynamics (cf. [4]), the basic restriction for the efficiency of heat engines.

Let us first set the list of postulates we impose for entanglement measure. So far, the rule of choosing some postulates and discarding others was an intuitive understanding of what entanglement is. Now, we would like to add a new rule: *Entanglement of distillation is a good measure*. Thus, we cannot accept a postulate that is not satisfied by $E_D$. This is reasonable because $E_D$ has a direct sense of the quantum capacity of the teleportation [15] channel constituted by the source producing bipartite systems. We will see that this rule will suppress some of the hitherto accepted postulates: This is the lesson given us by the existence of bound entangled states [16].

We split the postulates into the following three groups.

1. *Obvious postulates.*—(a) Non-negativity: $E(\varrho) \geq 0$; (b) vanishing on separable states: $E(\varrho) = 0$ if $\varrho$ is separable; (c) normalization: $E(|\psi_+\rangle\langle\psi_+|) = 1$, where $\psi_+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

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2. *Fundamental postulate: monotonicity under LQCC operations.*—(a) Monotonicity under local operation: If either of the parties sharing the pair in the state $\varrho$ performs the operation leading to state $\sigma_i$ with probability $p_i$, then the expected entanglement cannot increase

$$E(\varrho) \geq \sum_i p_i E(\sigma_i);$$

(b) convexity (monotonicity under discarding information):

$$E\left(\sum_i p_i \varrho_i\right) \leq \sum_i p_i E(\varrho_i).$$

3. *Asymptotic regime postulates.*—(a) Partial additivity:

$$E(\varrho^{\otimes n}) = nE(\varrho);$$

(b) continuity: If $\langle\psi^{\otimes n}|\varrho_n|\psi^{\otimes n}\rangle \to 1$ for $n \to \infty$, then

$$\frac{1}{n}\,|E(\psi^{\otimes n}) - E(\varrho_n)| \to 0,$$

where $\varrho_n$ is some joint state of $n$ pairs.

Let us now briefly discuss the considered postulates. In the first group, the postulate of normalization is to prevent us from the many trivial measures given by positive constant multiply of some measure $E$. The axiom 1(a) is indeed obvious (a separable state contains no entanglement). What, however, is not obvious is, Should we not require vanishing of $E$ if and *only* if the state is separable? The latter seems reasonable, because if the state is not separable, it contains entanglement that should be indicated by the entanglement measure. However, according to our rule, we should look at distillable entanglement. We can then see that the bound entangled states [16] are entangled, but have $E_D$ equal to zero. Thus we should accept entanglement measures that indicate no entanglement for some entangled states. This curiosity is due to the existence of different types of entanglement.

Let us now pass to the second group. The fundamental postulate, displaying the basic feature of entanglement (that creating entanglement requires *global* quantum interaction) was introduced in Refs. [1,2] and developed in Refs. [4–6]. It was put into the above, very convenient, form in Ref. [9]. Any function satisfying it must be invariant, under product unitary transformations and constant on separable states [9]. It also follows that, if a trace preserving map $\Lambda$ can be realized as a LQCC operation, then $E(\Lambda(\varrho)) \leq E(\varrho)$.

The postulates of the first and second groups are commonly accepted. The functions that satisfied them (without normalization axiom) have been called *entanglement monotones* [9].

Let us now discuss the last group of postulates, called "asymptotic regime ones" because they are necessary in the limit of large numbers of identically prepared entangled pairs, and can be discarded if a small number of pairs are considered. This asymptotic regime is extremely important as it is a natural regime both for the directly related theory of quantum channel capacity [2] and the recently developed "thermodynamics of entanglement" [4,13,17].

Partial additivity says that if we have a stationary, memoryless source, producing pairs in the state $\varrho$, then the entanglement content grows linearly with the number of pairs. A plausible argument to accept this postulate was given in Ref. [4] in the context of thermodynamical analogies. Plenio and Vedral [13] considered full additivity $E(\varrho \otimes \sigma) = E(\varrho) + E(\sigma)$ as a desired property. However, the effect of activation of bound entanglement [18] suggests that $E_D$ is not fully additive, so, according to our rule, we will not impose this stronger additivity.

Let us now pass to the last property. It states that, in the region close to the pure states, our measure is to behave regularly: If the joint state of large number pairs is close to the product of pure states, then the *densities* of entanglement (entanglement per pair) of both of the states should also be close to each other. This is a very weak form of the continuity exhibited, e.g., by von Neumann entropy that follows from Fannes inequality [19]. We do not require the latter, strong continuity, because we expect that entanglement of distillation can exhibit some peculiarities at the boundary of the set of bound entangled states. However, it can be seen that $E_D$ satisfies this weak continuity displayed as the last postulate of our list.

The continuity property as a potential postulate for entanglement measures was considered by Vidal [9] in the context of the problem of uniqueness of the entanglement measure for pure states. Namely, Popescu and Rohrlich [4], starting from thermodynamical analogies, argued that entanglement of formation (equal to entanglement of distillation for pure states [1]) is a unique measure, if one imposes additivity and monotonicity (and, of course, normalization). Later, many monotones different from $E_F$ on pure states were designed [5,6,8]. There was still no contradiction because they were not additive. However, Vidal constructed a set of monotone additives for pure states that still differed from $E_F$ for pure states [9]. He removed the contradiction by pointing out that the missing assumption was just the considered continuity. The completed-in-this-way *uniqueness theorem* states that a function satisfying the listed axioms must be equal to entanglement of formation on the pure states.

In the following we will show that the above theorem can be viewed as a special case of the general property of entanglement measures (in this paper, we will call the functions satisfying the list of postulates the entanglement measures). Before we state the theorem we need definitions of the entanglement of distillation and formation. We accept the following definitions.

$E_F$ is a regularized version of the original entanglement of formation $E_f$ [2] defined as follows. For pure states, $E_f$ is equal to entropy of entanglement, i.e., von Neumann entropy of either of the subsystems. For mixed states, it is given by

$$E_f(\varrho) = \min \sum_i p_i E_f(\psi_i), \quad \text{with } \varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{2}$$

where the minimum is taken over all possible decompositions of $\varrho$ (we call the decomposition realizing the minimum the optimal decomposition of $\varrho$). Now $E_F \equiv \lim_n E_f(\varrho^{\otimes n})/n$.

To define the distillable entanglement $E_D$ [2,7] (see Ref. [10] for justifying this definition) of the state $\varrho$, we consider distillation protocols $\mathcal{P}$ given by a sequence of trace-preserving, completely positive, superoperators $\Lambda_n$, that can be realized by using LQCC operations, and that map the state $\varrho^{\otimes n}$ of $n$ input pairs into a state $\sigma_n$ acting on the Hilbert space $\mathcal{H}_n^{\rm out} = \mathcal{H}_n \otimes \mathcal{H}_n$ with $\dim\mathcal{H}_n = d_n$. Define the maximally entangled state on the space $\mathcal{H} \otimes \mathcal{H}$ by

$$P_+(\mathcal{H}) = |\psi_+(\mathcal{H})\rangle\langle\psi_+(\mathcal{H})|, \qquad \psi_+(\mathcal{H}) = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle, \tag{3}$$

where $|i\rangle$ are basis vectors in $\mathcal{H}$, while $d = \dim\mathcal{H}$. Now $\mathcal{P}$ is the distillation protocol if, for high $n$, the final state approaches the above state $P_+$,

$$F \equiv \langle\psi_+(\mathcal{H}_n)|\sigma_n|\psi_+(\mathcal{H}_n)\rangle \to 1 \tag{4}$$

(i.e., the fidelity $F$ tends to 1). The asymptotic ratio $D_{\mathcal{P}}$ of distillation via protocol $\mathcal{P}$ is given by

$$D_{\mathcal{P}}(\varrho) \equiv \lim_{n\to\infty} \frac{\log_2 \dim\mathcal{H}_n}{n} \tag{5}$$

The distillable entanglement is defined by the maximum of $D_{\mathcal{P}}$ over all protocols,

$$E_D(\varrho) = \sup_{\mathcal{P}} D_{\mathcal{P}}. \tag{6}$$

Now, the main result of this paper is the following.

*Theorem 1.*—For any function $E$ satisfying the introduced postulates, and for any state $\varrho$, one has

$$E_D(\varrho) \le E(\varrho) \le E_F(\varrho). \tag{7}$$

*Remark.*—For pure states we have $E_D = E_F$; hence from the above inequality it follows that all measures are equal to $E_F$ in this case. This is compatible with the uniqueness theorem.

*Proof.*—Surprisingly enough, the proof is elementary. Both the left- and right-hand-side inequalities of the theorem are proved by the use of the same line of argumentation: (i) By definition, $E_D$ ($E_F$) is asymptotically constant during optimal distillation (formation) protocol; (ii) distillation (formation) protocol is an LQCC operation and cannot increase any entanglement measure; (iii) the final (initial) state is the pure one; (iv) for pure states all measures coincide by virtue of the uniqueness theorem.

It then easily follows that, if the given measure $E$ were, e.g., less than $E_D$, it would have to increase under optimal distillation protocol. We used here additivity, because formation and distillation protocols are collective operations (performed on $\varrho^{\otimes n}$). Continuity is needed, because we use the uniqueness theorem. By writing the above more formally in the case $E \le E_F$, we obtain

$$E(\varrho) = \frac{E(\varrho^{\otimes n})}{n} \le \frac{\sum_i p_i E(\psi_i)}{n} = \frac{\sum_i p_i E_f(\psi_i)}{n} = \frac{E_f(\varrho^{\otimes n})}{n} \stackrel{n\to\infty}{\longrightarrow} E_F(\varrho), \tag{8}$$

where we chose optimal decomposition of $\varrho^{\otimes n}$, so that the $\sum_i p_i E_f(\psi_i)$ is minimal and hence equal to $E_f(\varrho^{\otimes n})$ [20]. The first equality comes from additivity; the inequality is a consequence of monotonicity [more precisely—convexity, axiom 2(b)]. The next-to-last equality follows from the uniqueness theorem. We will skip the formal proof of the inequality $E_D \le E$, because in the following we prove formally a stronger result concerning bounds for entanglement of distillation.

Below we will show that the above, very transparent line of argumentation is a powerful tool, as it allows one to prove a very general theorem on the upper bounds of $E_D$.

*Theorem 2.*—Any function $B$ satisfying the conditions (a)–(c) below is an upper bound for entanglement of distillation: (a) Weak monotonicity: $B(\varrho) \ge B(\Lambda(\varrho))$ where $\Lambda$ is the trace-preserving superoperator realizable by means of LQCC operations. (b) Partial subadditivity: $B(\varrho^{\otimes n}) \le nB(\varrho)$. (c) Continuity for isotropic state $\varrho(F,d)$ [11,21]. The latter is of the form

$$\varrho(F,d) = pP_+(C^d) + (1-p)\frac{1}{d^2} I, \qquad 0 \le p \le 1 \tag{9}$$

with $\mathrm{Tr}[\varrho(F,d)P_+(C^d)] = F$. Suppose now that we have a sequence of isotropic states $\varrho(F_d,d)$, such that $F_d \to 1$ if $d \to \infty$. Then we require

$$\lim_{d\to\infty} \frac{1}{\log_2 d} B(\varrho(F_d,d)) \to 1. \tag{10}$$

*Remarks.*—(1) The above conditions are implied by our postulates for entanglement measures. Specifically, the condition (a) is implied by monotonicity; (b), by additivity; while the condition (c), by continuity plus additivity. (2) If instead of LQCC operations we take other class $C$ of operations, including one-way classical communication, the *mutatis mutandis* proof also applies [then the condition (a) would involve the class $C$].

*Proof.*—We will perform analogous evaluation as in formula (8) (now, however, we will not even use the uniqueness theorem). By subadditivity we have

$$B(\varrho) \ge \frac{1}{n} B(\varrho^{\otimes n}). \tag{11}$$

Since the only relevant parameters of the output of the process of distillation are the dimension of the output Hilbert space and fidelity $F$ (see the definition of distillable entanglement), we can consider distillation protocol ended by twirling [21] that results in an isotropic final state. By condition (a), distillation does not increase $B$, and hence

$$\frac{1}{n} B(\varrho^{\otimes n}) \geq \frac{1}{n} B(\varrho(F_{d_n}, d_n)) . \qquad (12)$$

Now, in the limit of large $n$, distillation protocol produces $F \rightarrow 1$ and $(\log_2 d_n)/n \rightarrow E_D(\varrho)$; hence by condition (c) the right-hand side of the inequality tends to $E_D(\varrho)$. Thus we obtain that $B(\varrho) \geq E_D(\varrho)$.

Using the above theorem, to find a bound for $E_D$, three things must be done: one should show that a chosen function satisfies the weak monotonicity, then check subadditivity, and calculate it for the isotropic state, to check the condition (c). Note that the weak monotonicity is indeed much easier to prove than full monotonicity, as given by postulate 2(a). Checking subadditivity, in contrast to additivity, is in many cases immediate: It in fact holds for all so-far-known entanglement monotones. Finally, the isotropic state is probably the easiest possible state to calculate the value of a given function. To illustrate the power of the result let us prove that relative entropy entanglement $E_r$ is bound for $E_D$. Subadditivity and weak monotonicity are immediate consequences of the properties of relative entropy used in the definition of $E_r$ (subadditivity proved in Ref. [5], weak monotonicity proved in Ref. [6]). The calculation of $E_r$ for the isotropic state is a little bit more involved, but by using high symmetry of the state it was found to be [11] $E_r(\varrho(F,d)) = \log_2 d + F \log_2 F + (1 - F) \log_2 \frac{1-F}{d-1}$. By evaluating this expression now for large $d$, we easily obtain that the condition (c) is satisfied. The proof applies without any change to the Rains bound [11].

In summary, we have presented two results. The first one has conceptual meaning leading to deeper understanding of the phenomenon of entanglement. It provides some synthetic overview of the domain of quantifying entanglement in the asymptotic regime. One of the possible applications of the result would be to reverse the direction of reasoning, and accept the condition $E_D \leq E \leq E_F$ as a preliminary test for a good candidate for entanglement measure. The second result presented in this paper is of direct practical use. We believe that it will make the search for strong bounds on $E_D$ much easier, especially in higher dimensions. Finally, we would like to stress that the results display the power of the fundamental principle of entanglement processing: the latter allows one not only to replace a complicated proof by a straightforward one, but also makes the argumentation very transparent from the physical point of view.

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