Astrophysical Jets as Exact Plasma Equilibria

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Two families of exact global solutions to the equations of plasma equilibrium are derived. The solutions model astrophysical jets and solar prominences and provide counterexamples to Parker's hypothesis.

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The equations of plasma equilibrium are

$$\mathrm{curl}\mathbf{B} \times \mathbf{B} = \mu\, \mathrm{grad}p, \qquad \mathrm{div}\mathbf{B} = 0. \qquad (1)$$

The search for the global plasma equilibria was going on during the past four decades since Eqs. (1) were first applied to the controlled thermonuclear fusion [1,2] and to the astrophysical problems [3]. However, up until now, all found exact solutions to Eqs. (1) which are not translationally invariant, either have singularities or unboundedly grow at infinity [4–6], or are not localized [7]. Such solutions have a very limited applicability in astrophysics.

The primary purpose of this paper is to report some new exact solutions of the plasma equilibrium equations that model astrophysical jets in the comoving frame of reference and solar prominences. Such equilibria have to be global; that means they have to satisfy the following physical conditions in the cylindrical coordinates r, ϕ, z: (a) The magnetic field $\mathbf{B}$ and pressure p are smooth and bounded in $\mathbb{R}^3$; (b) at $r \to \infty$, the magnetic field $\mathbf{B} \to 0$, the pressure $p \to p_1$; (c) all magnetic field lines are bounded in the radial variable r.

We derive two families of exact global axially symmetric plasma equilibria which depend on an arbitrary number of free parameters. The first one is defined in the whole Euclidean space $\mathbb{R}^3$ and the second one in the half-space $z \geq 0$. These exact plasma equilibria model the variety of magnetic fields observed in astrophysical jets and in solar corona. The asymptotic value of pressure p_1 in condition (b) is the average pressure in the astrophysical outflow or in solar coronal plasma. As usual, the gravitational force $-\rho\, \mathrm{grad}\Psi$ is included into the pressure gradient in (1), in the approximation of constant density ρ. The plasma equilibria are localized in the sense that the total magnetic energy in any layer $c_1 < z < c_2$ is finite. For the solutions of the second class, the total magnetic energy in the half-space $z \geq 0$ is finite.

The generic equilibrium solutions in $\mathbb{R}^3$ are quasiperiodic in variable z with $N - 1$ frequencies which are proportional to the numbers $1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{N-1}$ with the common factor $\sqrt{8\beta}$. It is proved that there are approximately $6N/\pi^2$ rationally independent square roots in the sequence. Therefore the magnetic field $\mathbf{B}$ is truly quasiperiodic in variable z.

The first of the exact solutions is the Gaussian distribution $\mathbf{B} = \exp(-\beta r^2)\hat{\mathbf{e}}_z$, $p = p_1 - \exp(-2\beta r^2)/2\mu$. The Gaussian distribution plays a crucial role in the derived global plasma equilibria for the solutions decrease at $r \to \infty$ as rapidly as $c_N \exp(-\beta r^2) r^{2N}$. For the exact nontranslationally invariant plasma equilibria, the magnetic surfaces are either cylinders or nested tori with circular magnetic axes. The distribution of these toroidal magnetic surfaces is quasiperiodic in variable z.

As a secondary purpose, I would like to point out that the two families of equilibrium solutions reported on provide a counterexample to the well-known Parker's hypothesis [8–10] that concerns the small perturbations of the z-invariant plasma equilibria and that has been around for 25 years. Parker writes [9], page 374: "Consider a magnetic field $B_i(x,y) + \epsilon b_i(x,y,z)$ in the neighborhood of the general equilibrium field $B_i(x,y)$"; and after a detailed study arrives at the conclusion on page 377:

"Thus, in the general case, we are led to the conclusion that the invariance $\partial b_i/\partial z = 0$ (14.51) is a necessary condition for equilibrium. Any field in which winding pattern changes along the field, so that (14.51) is excluded by the topology, cannot be in equilibrium."

We call this conclusion *Parker's hypothesis.* The absence of exact global solutions to Eqs. (1) made an independent verification of Parker's hypothesis impossible. Many consequences and generalizations were produced assuming that the hypothesis is true (see [11–17] and Parker's 1994 book [10]).

We present counterexamples to Parker's hypothesis which satisfy all Parker's conditions [9], pages 359–391, and the above physical conditions (a), (b), and (c). We construct a family of global z-invariant plasma equilibria; each Nth equilibrium possesses a $(2N-1)$-dimensional linear space of global perturbations. The most important feature of these exact solutions is that they do *depend* on variable z and, hence, they are *not* z-invariant.

The z-quasiperiodicity of the equilibrium solutions implies that they are very far not only from being z-invariant but even from being z-periodic. In view of the z-quasiperiodicity, the "winding pattern" of the magnetic field lines is continuously changing along the variable z and does not repeat.

To derive the exact plasma equilibria, we consider Eqs. (1) for an axially symmetric magnetic field $\mathbf{B}$ [1,2]:

$$\mathbf{B} = (\psi_z \hat{\mathbf{e}}_r - \psi_r \hat{\mathbf{e}}_z + I\hat{\mathbf{e}}_\phi)/r, \qquad (2)$$

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where $\psi(r,z)$ is the flux function and $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_z$, $\hat{\mathbf{e}}_\phi$ are the coordinate unit vectors. The plasma equilibrium Eqs. (1) are equivalent to the equations $I = I(\psi)$, $p = p(\psi)$, and the Grad-Shafranov equation [1,2]:

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + I(\psi)I'(\psi) + \mu r^2 p'(\psi) = 0\,. \tag{3}$$

As in [7,18], let $I(\psi)$ be linear, $I(\psi) = \alpha\psi$, and $p(\psi)$ be quadratic, $p(\psi) = p_1 - 2\beta^2\psi^2/\mu$, where $p_1 > (2\beta^2/\mu)\max[\psi^2(x,y,z)]$ and α and $\beta > 0$ are arbitrary constants. Equation (3) takes the form

$$\psi_{rr} - \psi_r/r + \psi_{zz} + \alpha^2\psi - 4\beta^2 r^2\psi = 0\,. \tag{4}$$

Using new variable $x = 2\beta r^2$ and separating variables in Eq. (4) by the substitution $\psi(r,z) = \exp(-x/2) \times P(x)T(z)$, we obtain

$$xP'' - xP' + (\alpha^2 + \lambda)P/8\beta = 0, \qquad T'' = \lambda T\,. \tag{5}$$

For $\lambda = -\omega^2$, we have $T(z) = a\cos(\omega z) + b\sin(\omega z)$. For the first Eq. (5), we are interested only in the polynomial solutions $P(x)$. Such solutions exist if and only if $(\alpha^2 - \omega^2)/8\beta = n$, where $n \geq 0$ is an integer. Hence, we find the finite spectrum of the admissible values of $\omega = \omega_n = \sqrt{\alpha^2 - 8\beta n}$, $n = 0, 1, \ldots, N$, $N = [\alpha^2/8\beta]$. The first Eq. (5) results in the form $xP'' - xP' + nP = 0$. Differentiating, we obtain $xL'' + (1 - x)L' + (n - 1)L = 0$, where $L(x) = P'(x)$. This equation defines the classical Laguerre polynomials $L_{n-1}(x)$. Hence, the polynomials $P(x)$ are primitive functions of the Laguerre polynomials. We denote them $L_n^*(x)$:

$$L_n^*(x) = \int_0^x L_{n-1}(t)\,dt = x + \sum_{k=1}^{n-1}\frac{(-1)^k(n-1)!}{k!\,(k+1)!\,(n-k-1)!}\,x^{k+1}. \tag{6}$$

Thus, we obtain the exact solutions to Eq. (4):

$$\psi_n(z,r) = \exp(-\beta r^2)L_n^*(2\beta r^2) \times [a_n\cos(\omega_n z) + b_n\sin(\omega_n z)]\,. \tag{7}$$

For any fixed constants α and β, the formulas (7) define N exact solutions to the linear Eq. (4). These solutions satisfy conditions (a) and (b) above.

Exact global plasma equilibria.—For $\alpha^2 = 8\beta N$, we get $\omega_N = 0$ and exact solution (7) takes the form

$$\psi_N(r) = a_N\exp(-\beta r^2)L_N^*(2\beta r^2)\,. \tag{8}$$

The corresponding magnetic field (2) is

$$\mathbf{B}_N = 2a_N e^{-\beta r^2}\{\beta[L_N^*(x) - 2L_N^{*\prime}(x)]\hat{\mathbf{e}}_z + \sqrt{2\beta N}\,L_N^*(x)\hat{\mathbf{e}}_\phi/r\}, \tag{9}$$

$p_N = p_1 - 2\beta^2\psi_N^2(r)/\mu$. This is a z-invariant global plasma equilibrium.

Taking a linear combination of the exact solutions (7) for $n = 1, \ldots, N-1$, and (8), we obtain the $(2N-1)$-dimensional linear space of exact solutions:

$$\psi(r,z) = e^{-\beta r^2}\Bigg\{a_N L_N^*(x) + \sum_{n=1}^{N-1} L_n^*(x) \times [a_n\cos(\omega_n z) + b_n\sin(\omega_n z)]\Bigg\}, \tag{10}$$

where $\omega_n = \sqrt{8\beta(N-n)}$, $x = 2\beta r^2$. For $N \geq 3$, the generic solutions (10) are quasiperiodic functions of z; for example, if $a_{N-1} \neq 0$, $a_{N-2} \neq 0$.

Poloidal projections of the magnetic field lines coincide with the level curves $\psi(r,z) = \text{const}$ [1,2]. Formula (10) implies that these curves approach the straight lines $r = \text{const}$ when $r \gg 1$ because its leading term is $-a_N(-2\beta r^2)^N\exp(-\beta r^2)/N!$. Hence, all magnetic field lines and all current lines are bounded in the radial variable r. Hence, the exact solutions (10) at $a_N \neq 0$ satisfy the physical conditions (a), (b), and (c).

The first three polynomials $L_n^*(x)$ (6) have the form $L_1^*(x) = x$, $L_2^* = x - x^2/2$, and $L_3^*(x) = x - x^2 + x^3/6$. Each polynomial $L_n^*(x)$ (6) has $n-1$ distinct positive roots and the root $x = 0$; its greatest root x_n is $> 2n - 3$.

Example 1: The astrophysical jets model.—Figure 1 shows the quasiperiodic level curves $\psi(r,z) = \text{const}$ for the exact solution (10) for $N = 3$, $\beta = 0.1$: $\psi(r,z) = e^{-\beta r^2}\{L_3^*(x) + 0.05\sin[2\sqrt{2\beta}\,z]L_2^*(x) + 0.05\sin[4\sqrt{\beta}\,(z-1)]L_1^*(x)\}$, $x = 2\beta r^2$. These curves are z-quasiperiodic for the frequencies $2\sqrt{2\beta}$ and $4\sqrt{\beta}$ are rationally independent. Rotating curves on Fig. 1 around the axis z, one obtains magnetic surfaces comprising continuous families of cylinders and nested tori. The innermost tori are circular magnetic axes.

Figure 2 represents the density of magnetic energy $B^2(x,y,z)/2\mu$ for the above plasma equilibrium for

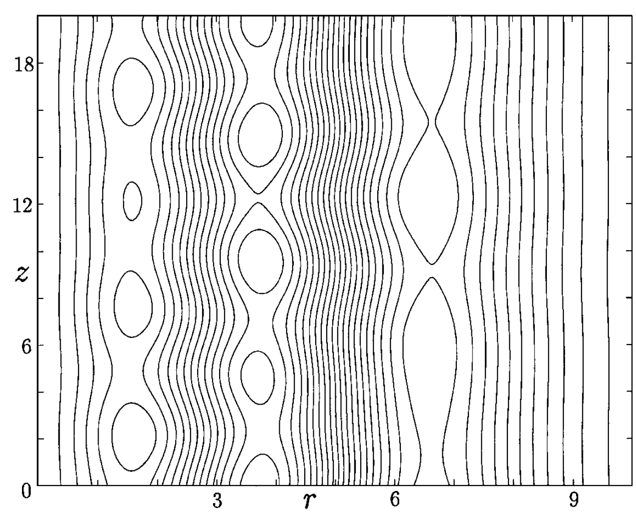

FIG. 1. Quasiperiodic magnetic field lines for the astrophysical jets model.

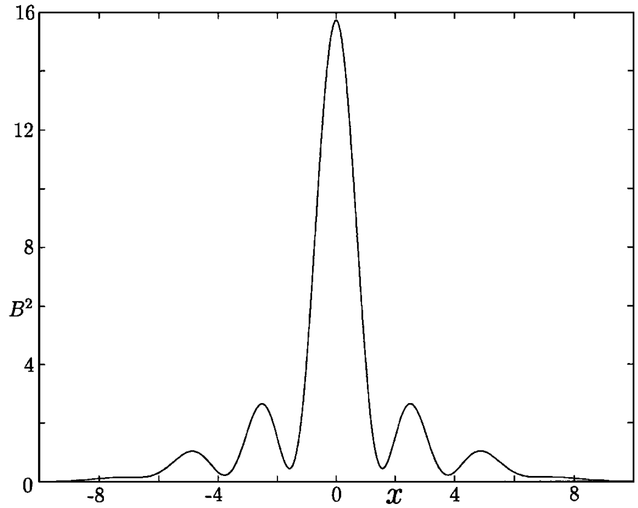

FIG. 2. Density of magnetic energy $B^2(x, 0, 0.5)/2\mu$.

$y = 0$, $z = 0.5$ and $\beta = 0.1$, $\mu = 0.5$. It is evident that $B^2(r, z) \neq 0$ everywhere and that $B^2(r, z) \to 0$ at $r \to \infty$. The magnetic energy is concentrated near the axis of symmetry. This property means that the above exact solution models an astrophysical jet in the comoving frame of reference.

Exact global plasma equilibria in the half-space $z \geq 0$.—Putting in Eq. (5) $\lambda = \kappa_n^2$, $\kappa_n = \sqrt{8\beta n - \alpha^2}$, $n = N, N + 1, \ldots, N = [\alpha^2/8\beta] + 1$, we find the exact solutions $P_n(x) = L_n^*(x)$, $T_n(z) = \exp(-\kappa_n z)$. Hence, we find that the linear Eq. (4) has the exact solutions

$$\psi(r, z) = e^{-\beta r^2}\left[\sum_{n=N}^{N+m} a_n \exp(-\sqrt{8\beta n - \alpha^2}\, z) \times L_n^*(2\beta r^2)\right], \tag{11}$$

where $m \geq 0$, $N \geq 0$ are arbitrary integers, α, $\beta > 0$, and a_n are arbitrary constants, $\alpha^2 \leq 8\beta N$. The solutions (11) are defined in the half-space $z \geq 0$ and rapidly tend to zero at $r \to \infty$ and at $z \to \infty$ if $\alpha^2 < 8\beta N$. The plasma pressure p has the form $p(\psi) = p_1 - 2\beta^2\psi^2/\mu$, where $p_1 > (2\beta^2/\mu)\max[\psi^2(x, y, z)]$, $z \geq 0$. The corresponding magnetic field (2), current $\mathbf{J} = \mathrm{curl}\mathbf{B}/\mu$ and pressure p have no singularities in the half-space $z \geq 0$. Hence, the exact solutions (11) satisfy the physical conditions (a) and (b). They satisfy also condition (c) if $\mathrm{sign} a_n = (-1)^n$. Hence, the flux functions (11) define the global plasma equilibria. For $\alpha^2 < 8\beta N$, the total magnetic energy of the exact global plasma equilibria (11) in the half-space $z \geq 0$ is finite. These equilibria model the solar prominences; the asymptotic value p_1 is the average pressure in the solar coronal plasma.

For $\alpha^2 = 8\beta N$, $x = 2\beta r^2$, the flux function (11) takes the form

$$\psi(r, z) = e^{-\beta r^2}\left\{a_N L_N^*(x) + \sum_{n=N+1}^{N+m} a_n \times \exp[-\sqrt{8\beta(n - N)}\, z] L_n^*(x)\right\}. \tag{12}$$

Example 2: The solar prominences model.—Figure 3 shows the magnetic field lines for the exact solution $\psi(r, z) = \exp(-\beta r^2)[\exp(-4\sqrt{\beta}\, z)L_2^*(x) - 0.5 \times \exp(-2\sqrt{6\beta}\, z)L_3^*(x)]$ for $\alpha = 0$, $\beta = 10$, $x = 2\beta r^2$. The corresponding magnetic field $\mathbf{B} = (\psi_z \hat{\mathbf{e}}_r - \psi_r \hat{\mathbf{e}}_z)/r$ is pure poloidal, the current $\mathbf{J} = 4\beta^2 r\psi\hat{\mathbf{e}}_\phi/\mu$ is circular. All magnetic field lines are anchored at $z = 0$ (the solar photosphere boundary).

Counterexamples to Parker's hypothesis.—For the z-invariant plasma equilibrium (8), the magnetic field $\mathbf{B}_N$ (9) is nonzero everywhere in the Euclidean space $\mathbb{R}^3$. Indeed, if component $B_\phi(x_1) = 0$, then $L_N^*(x_1) = 0$; hence, $L_N^{*\prime}(x_1) \neq 0$ because all roots of the polynomial $L_N^*(x)$ are simple and therefore component $B_z(x_1) \neq 0$. Hence, we have $|\mathbf{B}_N| > B(R)|a_N| > 0$ in any domain $0 \leq r \leq R$. Let magnetic field $\mathbf{B}$ correspond to the exact solution $\psi(r, z)$ (10). For $0 \leq r \leq R$, we have

$$\frac{|\mathbf{B} - \mathbf{B}_N|}{|\mathbf{B}_N|} < A_N \frac{C(R)}{B(R)}, \qquad A_N = \frac{1}{|a_N|}\sum_{n=1}^{N-1}(|a_n| + |b_n|). \tag{13}$$

Formulas (2), (8), and (10) imply that $|\mathbf{B} - \mathbf{B}_N|/|\mathbf{B}_N| \to 0$ at $r \to \infty$. Using this asymptotics and inequality (13), we derive $|\mathbf{B} - \mathbf{B}_N| \ll |\mathbf{B}_N|$ everywhere in $\mathbb{R}^3$ provided that $A_N \ll 1$.

Formulas (6) imply $|L_n^*(x)| < x^n/n!$. Hence, for $x > N$, we get $|L_1^*(x)| + \cdots + |L_{N-1}^*(x)| < x^{N-1}/(N - 2)!$. For the flux function ψ (10), we obtain $|\psi - \psi_N| < |a_N|A_N \exp(-\beta r^2)x^{N-1}/(N - 2)!$. For $x > x_N$, we have $L_N^*(x) > (x - x_N)^N/N!$, where x_N is the greatest root of $L_N^*(x)$. Hence, $|\psi_N| > |a_N|\exp(-\beta r^2)(x - x_N)^N/N!$, and we get $|\psi - \psi_N|/|\psi_N| < A_N N^2/[x(1 - x_N/x)^N]$. Hence, for $x > N^2 x_N$, we obtain $|\psi - \psi_N|/|\psi_N| < A_N$. The same inequality is true for the magnetic field. Thus, for $A_N \ll 1$, the perturbations (10) can be significant only

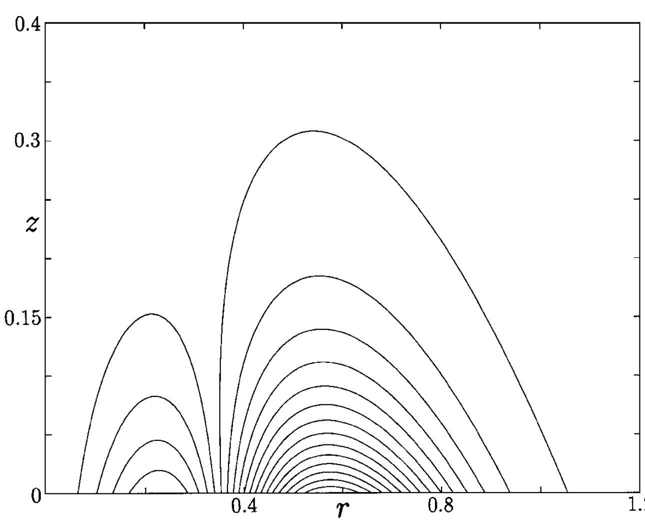

FIG. 3. Magnetic field lines for the solar prominences model.

for $x < N^2 x_N$. Substituting $x = 2\beta\ell^2$, we find for the length scale ℓ of the perturbations (10): $\ell \leq N\sqrt{x_N/2\beta}$.

The inequality $|\mathbf{B} - \mathbf{B}_N| \ll |\mathbf{B}_N|$ means that the plasma equilibria (10) at $A_N \ll 1$ are small perturbations in the whole Euclidean space $\mathbb{R}^3$ of the z-invariant equilibrium (8). Hence, we obtain that Parker's condition that "the local perturbation to the field is small compared to the total field" [9], page 361, is satisfied everywhere. Parker's condition that the length of the flux tube L is "large compared to the characteristic transverse scale of variation ℓ of the field" [9], page 362, is satisfied because $\ell \leq N\sqrt{x_N/2\beta}$ and the flux tube length L can be taken arbitrarily large for the z-invariant equilibrium (8). Hence, $L \gg \ell$. Parker's condition that "the magnetic field is analytic in its deviation ϵ from the invariant field $B_i(x,y)$" [9], page 378, is satisfied because the exact solutions (10) are linear functions of small parameters $a_1, b_1, \ldots, a_{N-1}, b_{N-1}$. All perturbations (10) are *not* z-invariant. Hence, the plasma equilibria (10) are counterexamples to Parker's hypothesis.

In the half-space $z \geq 0$, the plasma equilibria (12) are small perturbations of the z-invariant equilibria (8) at $A_N = (|a_{N+1}| + \cdots + |a_{N+m}|)/|a_N| \ll 1$. The above quoted Parker's conditions are satisfied at $A_N \ll 1$. All perturbations (12) are *not* z-invariant. Hence, the exact solutions (12) in the half-space $z \geq 0$ provide counterexamples to Parker's hypothesis.

One of the origins of the discrepancy with Parker's results.—In his book [9], Parker writes (page 369): "We suppose for convenience that, although $B_z(x,y)$ may vary widely, it does not vanish and change sign" and after a study arrives at the statement: "The result can be written

$$\frac{\partial}{\partial x}\frac{1}{B_z^2}\frac{\partial\Psi}{\partial x} + \frac{\partial}{\partial y}\frac{1}{B_z^2}\frac{\partial\Psi}{\partial y} + \frac{\partial}{\partial z}\frac{1}{B_z^2}\frac{\partial\Psi}{\partial z} = 0\,.$$

This form is totally elliptic. In an infinite space its only bounded solutions are constants, $\Psi = C$."

The proof of Parker's hypothesis on pages 369–370 [9], and the proof of its generalization for magnetohydrodynamics [14], page 837, Eq. (62), are based on this statement. We present a counterexample to this statement also. Let $F(u)$ be any smooth function such that $F(u) > 0$ for all u and $\int_{-\infty}^{+\infty} F^2(u)\,du < C_1$. Let $h(x,y)$ be any harmonic function: $\partial^2 h/\partial x^2 + \partial^2 h/\partial y^2 = 0$. We define

$$\begin{aligned} B_z(x,y) &= F[h(x,y)] > 0\,, \\ \Psi(x,y,z) &= \int_{-\infty}^{h(x,y)} F^2(u)\,du\,. \end{aligned} \tag{14}$$

A direct verification proves that function $\Psi(x,y,z)$ does satisfy the above elliptic equation, is bounded in $\mathbb{R}^3$, $0 < \Psi(x,y,z) < C_1$, and *nonconstant.*

Conclusion.—We have presented two families, (10) and (11), of the exact axially symmetric solutions to the plasma equilibrium Eqs. (1). The equilibrium solutions depend on an arbitrary number of parameters which can be adjusted so that the topology of the magnetic field becomes arbitrarily complex. The fields are a combination of nested cylindrical magnetic surfaces and nested toroidal surfaces. The innermost tori are circular magnetic axes that are linked with the neighboring closed magnetic field lines. The generic solutions in the family (10) are quasiperiodic in z, which implies that the magnetic field lines never repeat in the z direction, but can have a structure arbitrarily close to the initial data. Their "winding pattern" changes continuously with z, and does not repeat.

The equilibrium solutions (10), (11), and (12) are global, and everywhere smooth. There are no discontinuities, and there are no current sheets. These solutions model the variety of magnetic field phenomena that have been observed in the astrophysical jets (in the comoving frame of reference) and in the solar coronal plasma. The quasiperiodic behavior, which is ergodic in z, is the best available analytical description of the spatial distribution of the magnetic field and current in astrophysical jets.

Each of these families, (10) and (12), form counterexamples to Parker's hypothesis which was formulated over 25 years ago, and has been the starting point of a large body of work that has been published since that time. The most recent papers which use Parker's hypothesis in an essential way are the 1996 paper [12] and Parker's 1994 book [10].

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