Prediction of Spatiotemporal Time Series Based on Reconstructed Local States

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(Received 16 June 1999)

Spatiotemporal time series are analyzed and predicted using reconstructed local states. As numerical examples the evolution of a Kuramoto-Sivashinsky equation and a coupled map lattice are predicted from previously sampled data.

PACS numbers: 05.45.Tp

The starting points for most analysis methods used in *nonlinear time series analysis* are measurements of a single observable of the system of interest [1]. Many interesting dynamical systems, however, are spatially extended and thus any description using only a few local or global observables may be incomplete. On the other hand, technical devices for recording, storing, and analyzing sufficiently long spatiotemporal time series (STTS) are widely available now. We therefore address in this Letter the analysis of (large) data sets from spatially extended systems using an approach for local state space reconstructions.

Let $\{\mathbf{s}^n\}$ be the temporal sequence of spatial patterns (snapshots) of the spatiotemporal evolution with $n = 1, \ldots, N$. Each pattern $\mathbf{s}^n$ consists of M elements (pixels) and is therefore represented as an M dimensional vector with elements s_m^n $(m = 1, \ldots, M)$. Most of the techniques applied so far to STTS are based on decompositions into spatial modes that constitute orthogonal bases in a high dimensional vector space. Examples for such basic modes are the well-known Fourier modes, wavelets, or bases that are computed from the data, for example, by means of a Karhunen-Loève transformation (KLT) (also called proper orthogonal decomposition, singular value decomposition, empirical orthogonal eigenfunctions) [2]. The STTS $\{\mathbf{s}^n\}$ is projected onto these spatial modes and the resulting projection coefficients contain all essential information about the dynamics. In particular, for the KLT very often only a few coefficients are significantly different from zero and therefore the underlying process can be described in a low dimensional subspace (see Ref. [3] for a successful application of this approach). However, not always can a linear decomposition into basic modes yield a low dimensional description of the data, even in cases where the STTS is governed by a low dimensional attractor [4]. Modeling a STTS can also be viewed as a system identification task and only recently a promising approach for extracting partial differential equations (PDEs) from data has been presented by Voss *et al.* [5].

An alternative to decomposition into global linear modes or identifying global nonlinear models is the construction of local models that can be used for describing spatially extended systems whose dynamics is governed by *local* (inter)actions.

In order to explain the main idea as clearly as possible we assume in the following that the local dynamics is the same at different points in space (except for boundary effects). This assumption of homogeneity is of course not fulfilled by typical real world experimental systems but it may be viewed as a good approximation for many cases. A generalization for nonhomogeneous systems is possible and will be discussed briefly in the conclusion.

As a prerequisite we assume that the (local) state of the system in a small region of space may be represented by a vector $\mathbf{x}$. This reconstruction of a local state space can be done in different ways and the basic idea was (to our knowledge) suggested for the first time by Kaneko in Ref. [6]. Later Rubin [7] used a similar approach for characterizing dynamic and static patterns, and recently Orstavik and Stark [8] used spatiotemporal embedding techniques for cross prediction of coupled map lattices. These authors considered the local spatial region of interest as a low dimensional system that interacts with the global lattice via its boundaries. From this point of view the high dimensional global system is viewed as a stochastic forcing of the local low dimensional system and one can expect satisfying results (e.g., small prediction errors) only if the coupling and the influence of the stochastic components are weak. In contrast to this interpretation we shall in the following assume that a local state exists in a unique and deterministic sense that allows in principle exact predictions (without stochastic components). This conjecture is motivated by the case of coupled map lattices where it can easily be verified. Numerical simulations with coupled oscillators and PDEs show, however, that it seems to be correct also for other classes of systems. We want to stress that the latter is a necessary condition for the reconstruction techniques to be useful in time series analysis of real world data, because as soon as the local states are successfully (re)constructed they can be used for subsequent analysis or, as in the example given below, for predicting the underlying dynamics.

In the following we will discuss only the case of a one dimensional spatial pattern. Generalizations for higher dimensional cases are straightforward.

Reconstruction of local states. —The entire STTS may be represented by a $N \times M$ matrix S as shown in Fig. 1. The state of the system at position m and time n is

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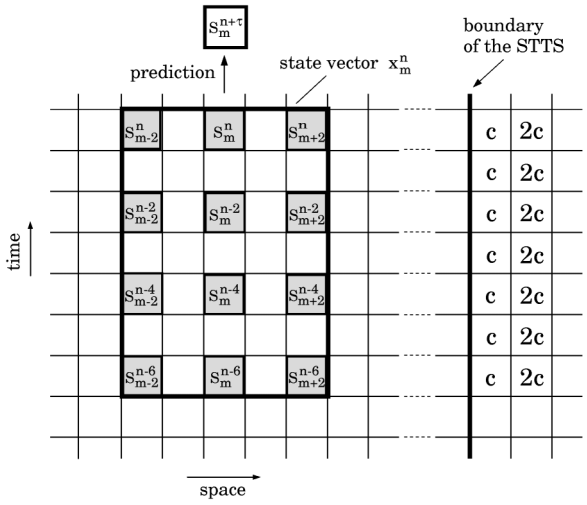

FIG. 1. Local reconstruction of states from regions of the spatiotemporal time series $S = \{s_m^n\}_{m=1,\ldots,M}^{n=1,\ldots,N}$ and prediction of the future value of the center element $s_m^{n+\tau}$. The constants c, $2c, \ldots$ on the right-hand side indicate the extension of the STTS beyond its boundary.

reconstructed in analogy to the delay embedding of the scalar time series [9]. Here we use the center element s_m^n, some of its neighbors, and the corresponding values in the past to construct the *state vector*

$$\mathbf{x}_m^n = (s_{m-IK}^n, \ldots, s_m^n, \ldots, s_{m+IK}^n, \ldots, s_{m-IK}^{n-JL}, \ldots, s_m^{n-JL}, \ldots, s_{m+IK}^{n-JL}), \quad (1)$$

where I is the number of spatial neighbors, J is the number of temporal neighbors (in the past), and K is the spatial shift which has a similar meaning as the time delay L (time lag) known from the delay embedding of the scalar time series. This construction is visualized in Fig. 1 for $I = 1$, $J = 3$, $K = 2$, and $L = 2$. The dimension d of the state vector $\mathbf{x}_m^n \in R^d$ is given by $d = (J + 1)(1 + 2I)$.

Boundary conditions.—Two boundaries have to be taken into account: (i) the boundary of the region from which data have been collected (or in other words, the boundary of the STTS) and (ii) the physical boundary of the system under investigation that may influence the dynamics close to that boundary. Because of the boundary of the STTS, components of the local state vector $\mathbf{x}_m^n$ in (1) are "missing" when trying to construct states close to this boundary. This problem can be overcome by extending the STTS in its spatial direction with numbers $-c, -2c, -3c, \ldots$ to the left and $c, 2c, 3c, \ldots$ to the right as indicated in Fig. 1. The parameter c has to be chosen larger than the largest value of the STTS. Using this construction all states close to the boundary of the STTS are located in different subspaces of the reconstruction space R^d and can thus be treated separately by subsequent algorithms. The effective dimension of the reconstructed boundary states is lower than d. Therefore, in principle different values of the reconstruction parameters I, J, K, and L could (and should) be used for the boundary to improve the reconstruction (and the results of any subsequent application of it). Physical boundary effects of the underlying system can be taken into account using a *penalty function*

$$w(\tilde{x}) = a\tilde{x}^b \quad (2)$$

with $-1 \le \tilde{x} \le 1$ being a normalized spatial coordinate (running from the left to the right boundary) and a and b are some free parameters. The basic idea of this approach is that states with a similar location with respect to the boundaries are governed by a similar dynamics and the additional quantity $w(x)$ provides this information about the dynamics for subsequent analysis and modeling. In this way one may also take into account inhomogeneous dynamics that changes not only near the boundaries but also in the interior of the spatial domain.

Nonlinear prediction.—As an application of the local state reconstruction we consider here the prediction of the future value of the central element $s_m^{n+\tau}$ "in front" of the reconstruction region (see Fig. 1) where τ gives the prediction time interval. The states are implemented using the above introduced formal extension of the STTS near boundaries. Then a training set of states $\mathcal{A} = \{\mathbf{x}_m^n\}$ is derived from N_{train} successive samples $\mathbf{s}^n$ of the STTS. For these states the preimage-image relation $\mathbf{x}_m^n \mapsto s_m^{n+\tau}$ is known and is assumed to represent a nonlinear map $f\colon R^d \to R$. In order to determine the future value $s_m^{n+\tau}$ of an element s_m^n of the STTS the corresponding state $\mathbf{x}_m^n$ is reconstructed. Then the nearest neighbor $\mathbf{x}_j^i$ of $\mathbf{x}_m^n$ is selected from the training set $\mathcal{A}$. Using the indices (i, j) of the nearest neighbor the underlying map f is locally approximated by the future value $p_m^{n+\tau} = s_j^{i+\tau}$ of s_j^i.

Of course, locally linear or nonlinear maps may also be used to approximate the dynamics f. Furthermore, predictions over longer periods of time ($\tau > 1$) can be computed as a single large step or iteratively by concatenating steps with $\tau = 1$.

Numerical examples.—We shall now present two numerical examples in order to illustrate the above described schemes for reconstructing local states and iterative prediction, i.e., we used the predicted values p_m^{n+1} to replace the original elements s_m^{n+1} after each step.

The first STTS is generated by a lattice of $M = 100$ coupled Hénon maps:

$$u_m^{n+1} = 1 - 1.45\left[\frac{1}{2}u_m^n + \frac{u_{m-1}^n + u_{m+1}^n}{4}\right]^2 + 0.3v_m^n, \quad (3)$$

$$v_m^{n+1} = u_m^n, \quad (4)$$

with fixed boundary conditions $u_1^n = u_M^n = 0.5$ and $v_1^n = v_M^n = 0$. Coupled map lattices are fundamental models for spatiotemporal chaos and structure formation [10]. They are discrete in space and time with continuous variables and correspond exactly to the structure of our prediction

ansatz. Therefore, they are ideal candidates for motivating, illustrating, and testing the reconstruction and prediction methods based on local states. For this example, the local states are reconstructed using $I = J = K = L = 1$ and states close to the boundary are (spatially) extended by constant values with $c = 20$. The prediction is performed iteratively ($\tau = 1$) based on a training set of length $N = 100$. The parameters of the penalty function (2) equal $a = 1 = b$ and the quantity w is included in the state vectors as an additional coordinate in order to bias the selection of nearest neighbors with similar distances to the boundary. Figure 2a shows the spatiotemporal evolution of the STTS which is to predicted, Fig. 2b the result of the prediction, and Fig. 2c the prediction error. Even in spatial regions with complex dynamics prediction is possible over several time steps although a training set of only 100 iterations has been used and the underlying dynamics is very high dimensional. As can be seen, stable (i.e., almost periodic) local patterns occur that seem to prevent the propagation of (small) errors that are amplified in other neighboring areas. In this sense the predictability of a spatiotemporal data set can be very different depending on the formation of structures [10].

The main reason for the good predictability of the data from the coupled Hénon maps is the fact that for this example the structure of the prediction scheme coincides very well with the structure of the equations used for generating the STTS. That this is not a necessary condition shows our second example where the STTS is generated using the *Kuramoto-Sivashinsky* (KS) equation [11,12]

$$u_t = -2uu_x - u_{xx} - u_{xxxx} \tag{5}$$

in the interval $[0, L]$ with $u = u_x = 0$ at the boundaries $x = 0$ and $x = L = 200$. The spatiotemporal dynamics of this system is governed by a hyperchaotic attractor with a Lyapunov dimension of $D_L \approx 43$. For predicting the dynamics of this PDE, states close to boundaries have been extended using $c = 20$ and neighbors are selected using the penalty function (2) with $a = 1$ and $b = 5$. The prediction is again performed iteratively. Figure 3a shows the spatiotemporal evolution of the KS equation in the time interval that is used as a training set for the prediction of the test data given in Fig. 3b. Figure 3c shows the results of a iterative prediction ($\tau = 1$) based on a reconstruction of local states with $I = 9$, $K = 1$, $J = 0$, and $L = 1$. As can be seen in Figs. 3b and 3c the essential features of the time evolution are correctly predicted including the splitting and merging of structures.

Similar to the case of delay embedding of scalar time series [1] the choice of proper embedding parameters is crucial for successful applications. Practically one may proceed in two steps. First the values for the spatial shift K and the temporal delay L are estimated using the (averaged) mutual information H [1] of spatial or temporal neighbors in the STTS as a function of K or L, respectively, in order to minimize the redundancy of the components of the local state vectors. For the data from the coupled Hénon maps redundancy of spatial and temporal neighbors decayed sufficiently already for $K = 1$ and $L = 1$, respectively. In the case of the Kuramoto-Sivashinsky data the H-K curve possesses a local minimum at $K = 4$ whereas the H-L curve has no pronounced minimum but decays

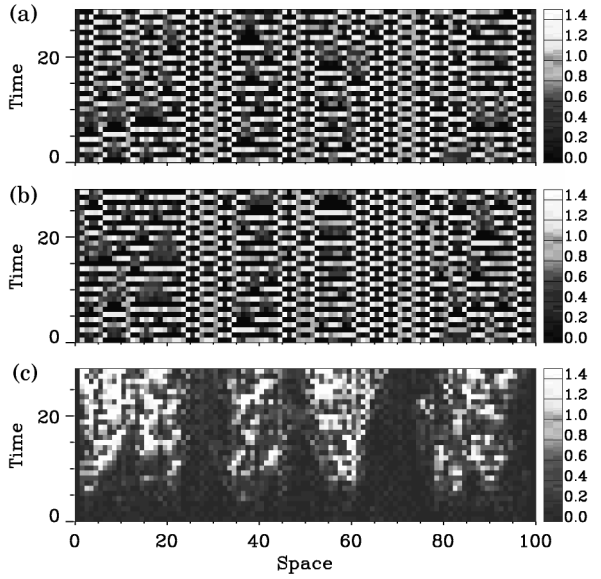

FIG. 2. Spatiotemporal time series generated by the Hénon map lattice (3). Values of the variable u_m^n are plotted gray scaled vs space m and time n. (a) Original time series to be predicted (not included in the training set). (b) Predicted time series. (c) Difference between the original time series (a) and the predicted data (b).

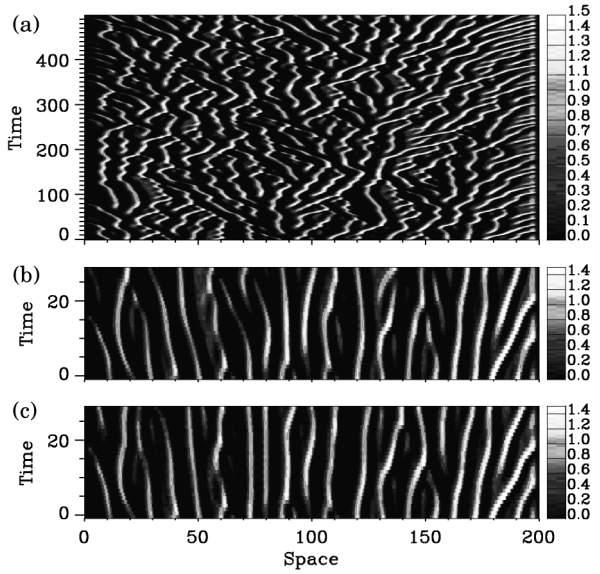

FIG. 3. Spatiotemporal time series generated by the Kuramoto-Sivashinsky equation (5). Values of the variable $u(x, t)$ are plotted gray scaled vs space x and time t. (a) Training set. (b) Original time series to be predicted. (c) Predicted time series.

sufficiently for $L = 1$, indicating a low redundancy of temporal neighbors for this value of the lag. These values for K and L may be used then to determine the necessary number of spatial I and temporal J neighbors (i.e., the dimension) of the reconstruction by increasing I and J until the prediction error decreases significantly. Practically, however, values of K and L larger than 1 may lead to decoupled dynamics of different subgrids and mesh-drift instabilities known from numerical schemes for solving PDEs [13]. To avoid this instability one may (i) add a numerical viscosity term to the prediction scheme [13], (ii) use an overlap-add approach for forecasting [14], or (iii) construct local state vectors by subjecting local regions in space-time to Karhunen-Loève transformations with subsequent projections onto a small number of dominating modes [15]. Since all these "cures" have particular drawbacks we suggest to sample the local states with $K = 1 = L$ and then use a nearest-neighbor search algorithm in the resulting high dimensional state space R^d that exploits the (expected) low dimensionality of the set of state vectors [16].

The numerical examples show that the local reconstruction of states is a powerful method for predicting spatiotemporal time series. It may also serve as a starting point for deriving a mathematical model of the underlying dynamics and subsequent bifurcation analysis. The schemes discussed in this paper may be generalized in different directions. The dimension of the reconstructed states can be reduced if the STTS stems from a dynamical system that possesses additional spatial symmetries that can be exploited when constructing the state vectors. For the one dimensional case we may, for example, assume that the system is invariant with respect to spatial reflection. If the process generating the STTS is not spatially homogeneous one may just add to the dimension d of the reconstruction space the number d_S of spatial dimensions of the problem (i.e., $d_S = 1, 2,$ or 3) and work then in the extended $d + d_S$ dimensional space [i.e., using the penalty function (2) with $a = 1 = b$]. Of course, in that case longer training sets will be necessary in general. The reconstruction and prediction methods also worked well for data that were not sampled simultaneously but (slowly) scanned spatially as is the case in many experimental measurements of extended systems. Another modification of the scheme proposed concerns the selection of spatiotemporal neighbors of the center element s_m^n for reconstructing the local state $\mathbf{x}_m^n$. At this point one may take into account the fact that any physical information spreads with some maximum speed. Instead of using a rectangular region of the matrix S a triangle (i.e., a "light cone") may be more efficient for reconstructing local states. This feature of locality may also be viewed as a motivation for the concept of local states and seems to be important for a rigorous mathematical justification of the presented method.

The authors acknowledge support by the *Bundesministerium für Bildung, Wissenschaft, Forschung und Technologie* (Grant No. 13 N 7038/9) and thank L. Junge for providing the Kuramoto-Sivashinsky data and W. Lauterborn for stimulating cooperation and continuous support.

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