Partial Dynamical Symmetry in a Fermion System

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The relevance of the partial dynamical symmetry concept for an interacting fermion system is demonstrated. Hamiltonians with partial SU(3) symmetry are presented in the framework of the symplectic shell model of nuclei and shown to be closely related to the quadrupole-quadrupole interaction. Implications are discussed for the deformed light nucleus ^{20}Ne.

Symmetries play an important role in dynamical systems. They provide labels for the classification of states, determine selection rules, and simplify the relevant Hamiltonian matrices. Algebraic, symmetry-based models offer significant simplifications when the Hamiltonian under consideration commutes with all the generators of a particular group ("exact symmetry") or when it is written in terms of the Casimir operators of a chain of nested groups ("dynamical symmetry") [1]. In both cases, basis states belonging to inequivalent irreducible representations (irreps) of the relevant groups do not mix, the Hamiltonian matrix has block structure, and all properties of the system can be expressed in closed form. An exact or dynamical symmetry not only facilitates the numerical treatment of the Hamiltonian but also its interpretation, and thus provides considerable insight into the physics of a given system.

Naturally, the application of exact or dynamical symmetries to realistic situations has its limitations. Usually, the assumed symmetry is only approximately fulfilled, and imposing certain symmetry requirements on the Hamiltonian might result in constraints which are too severe and incompatible with experimentally observed features of the system. The standard approach in such situations is to break the symmetry. Partial dynamical symmetry (PDS) [2] corresponds to a particular symmetry-breaking for which the Hamiltonian is not invariant under the symmetry group and, hence, various irreps are mixed in its eigenstates; yet it possesses a subset of "special" solvable states which respect the symmetry. This new scheme has recently been introduced in bosonic systems and has been applied to the spectroscopy of deformed nuclei [3] and to the study of mixed systems with coexisting regularity and chaos [4]. It is the purpose of this Letter to demonstrate the relevance of the partial dynamical symmetry concept to fermion systems. More specifically, in the framework of the symplectic shell model of nuclei [5], we will prove the existence of a family of fermionic Hamiltonians with partial SU(3) symmetry. The PDS Hamiltonians are rotationally invariant and closely related to the quadrupole-quadrupole interaction; hence, our study will shed new light on this important interaction. We will compare the spectra and eigenstates of the quadrupole-quadrupole and PDS Hamiltonians for the deformed light nucleus ^{20}Ne.

The quadrupole-quadrupole interaction is an important ingredient in models that aim at reproducing quadrupole collective properties of nuclei. A model which is able to fully accommodate the action of the collective quadrupole operator, $Q_{2m} = \sqrt{\frac{16\pi}{5}} \sum_s r_s^2 Y_{2m}(\hat{r}_s)$, is the symplectic shell model (SSM), an algebraic scheme which respects the Pauli exclusion principle [5]. In the SSM, this operator takes the form $Q_{2m} = \sqrt{3}(\hat{C}_{2m}^{(11)} + \hat{A}_{2m}^{(20)} + \hat{B}_{2m}^{(02)})$, where $\hat{A}_{lm}^{(20)}$, $\hat{B}_{lm}^{(02)}$, and $\hat{C}_{lm}^{(11)}$ are symplectic generators with good SU(3) [superscript (λ, μ)] and SO(3) [subscript l, m] tensorial properties. The $\hat{A}_{lm}^{(20)}$ ($\hat{B}_{lm}^{(02)}$), $l = 0$ or 2, create (annihilate) $2\hbar\omega$ excitations in the system. The $\hat{C}_{lm}^{(11)}$, $l = 1$ or 2, generate a SU(3) subgroup and act only *within* one harmonic oscillator (h.o.) shell ($\sqrt{3}\,\hat{C}_{2m}^{(11)} = Q_{2m}^E$, the symmetrized quadrupole operator of Elliott, which does not couple different h.o. shells [6], and $\hat{C}_{1m}^{(11)} = \hat{L}_m$, the orbital angular momentum operator). A fermion realization of these generators is given in [7].

A basis for the symplectic model is generated by applying symmetrically coupled products of the $2\hbar\omega$ raising operator $\hat{A}^{(20)}$ with itself to the usual $0\hbar\omega$ many-particle shell-model states. Each $0\hbar\omega$ starting configuration is characterized by the distribution of oscillator quanta into the three Cartesian directions, $\{\sigma_1, \sigma_2, \sigma_3\}$ ($\sigma_1 \geq \sigma_2 \geq \sigma_3$), or, equivalently, by its $U(1) \times SU(3)$ quantum numbers $N_\sigma(\lambda_\sigma, \mu_\sigma)$. Here $\lambda_\sigma = \sigma_1 - \sigma_2$, $\mu_\sigma = \sigma_2 - \sigma_3$ are the Elliott SU(3) labels, and $N_\sigma = \sigma_1 + \sigma_2 + \sigma_3$ is related to the eigenvalue of the oscillator number operator. The product of $N/2$ raising operators $\hat{A}^{(20)}$ generates $N\hbar\omega$ excitations for each starting irrep $N_\sigma(\lambda_\sigma, \mu_\sigma)$. Each such product operator $\mathcal{P}^{N(\lambda_n, \mu_n)}$, labeled according to its SU(3) content, (λ_n, μ_n), is coupled with $|N_\sigma(\lambda_\sigma, \mu_\sigma)\rangle$ to good SU(3) symmetry $\rho(\lambda, \mu)$, with ρ denoting the multiplicity of the coupling $(\lambda_n, \mu_n) \otimes (\lambda_\sigma, \mu_\sigma)$. The quanta distribution in the resulting state is given by $\{\omega_1, \omega_2, \omega_3\}$, with $N_\sigma + N = \omega_1 + \omega_2 + \omega_3$, $\omega_1 \geq \omega_2 \geq \omega_3$, and $\lambda = \omega_1 - \omega_2$, $\mu = \omega_2 - \omega_3$. The basis state construction is schematically illustrated in Fig. 1 for a typical Elliott starting state with $(\lambda_\sigma, \mu_\sigma) = (\lambda, 0)$. ^{20}Ne, for instance, has $N_\sigma = 48.5$ (after removal of the center-of-mass contribution) and $(\lambda_\sigma, \mu_\sigma) = (8, 0)$

[5,8]. To complete the basis state labeling, additional quantum numbers $\alpha = \kappa LM$ are required, where L denotes the angular momentum with projection M, and κ is a multiplicity index, which enumerates multiple occurrences of a particular L value in the SU(3) irrep (λ, μ) from 1 to $\kappa_L^{\max}(\lambda, \mu) = [(\lambda + \mu + 2 - L)/2] - [(\lambda + 1 - L)/2] - [(\mu + 1 - L)/2]$, where $[\cdots]$ is the greatest non-negative integer function [9]. The group chain corresponding to this labeling scheme is $\mathrm{Sp}(6, R) \supset \mathrm{SU}(3) \supset \mathrm{SO}(3)$ which defines a dynamical symmetry basis.

The quadrupole-quadrupole interaction connects h.o. states differing in energy by $0\hbar\omega$, $\pm 2\hbar\omega$, and $\pm 4\hbar\omega$, and may be written as

$$Q_2 \cdot Q_2 = 9\hat{C}_{\mathrm{SU3}} - 3\hat{C}_{\mathrm{Sp6}} + \hat{H}_0^2 - 2\hat{H}_0 - 3\hat{L}^2 - 6\hat{A}_0\hat{B}_0$$
$$+ \{\text{terms coupling different h.o. shells}\}, \quad (1)$$

where $\hat{C}_{\mathrm{SU3}}$ and $\hat{C}_{\mathrm{Sp6}}$ are the quadratic Casimir invariants of SU(3) and Sp(6,R) with eigenvalues $2(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu)/3$ and $2(\lambda_\sigma^2 + \mu_\sigma^2 + \lambda_\sigma\mu_\sigma + 3\lambda_\sigma + 3\mu_\sigma)/3 + N_\sigma^2/3 - 4N_\sigma$, respectively. These operators, as well as the h.o. $\hat{H}_0$ and $\hat{L}^2$ terms, are diagonal in the dynamical symmetry basis. Unlike the Elliott quadrupole-quadrupole interaction, $Q_2^E \cdot Q_2^E = 6\hat{C}_{\mathrm{SU3}} - 3\hat{L}^2$, the $Q_2 \cdot Q_2$ interaction of Eq. (1) breaks SU(3) symmetry within each h.o. shell since the term $\hat{A}_0\hat{B}_0 \equiv \hat{A}_0^{(20)}\hat{B}_0^{(02)} = (\{\hat{A} \times \hat{B}\}_0^{(00)} - \sqrt{5}\{\hat{A} \times \hat{B}\}_0^{(22)})/\sqrt{6}$ mixes different SU(3) irreps. In order to study the action of $Q_2 \cdot Q_2$ within such a shell, we consider the following family of Hamiltonians:

$$H(\beta_0, \beta_2) = \beta_0\hat{A}_0\hat{B}_0 + \beta_2\hat{A}_2 \cdot \hat{B}_2$$
$$= \frac{\beta_2}{18}(9\hat{C}_{\mathrm{SU3}} - 9\hat{C}_{\mathrm{Sp6}} + 3\hat{H}_0^2 - 36\hat{H}_0)$$
$$+ (\beta_0 - \beta_2)\hat{A}_0\hat{B}_0. \quad (2)$$

For $\beta_0 = \beta_2$, one recovers the dynamical symmetry, and, with the special choice $\beta_0 = 12$, $\beta_2 = 18$, one obtains $Q_2 \cdot Q_2 = H(\beta_0 = 12, \beta_2 = 18) + \mathrm{const}(N) - 3\hat{L}^2 + \text{terms coupling different shells}$, where $\mathrm{const}(N)$ is constant for a given h.o. $N\hbar\omega$ excitation.
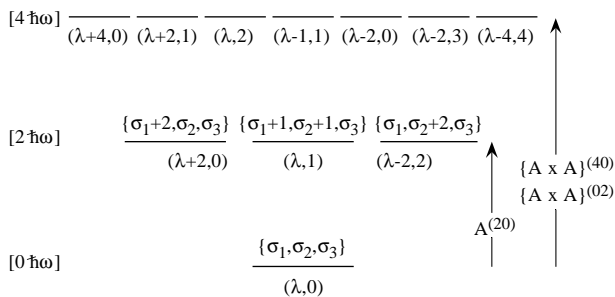

FIG. 1. Basis construction in the symplectic model. SU(3)-coupled products of the raising operator $\hat{A}^{(20)}$ with itself act on an Elliott starting state with $(\lambda_\sigma, \mu_\sigma) = (\lambda, 0)$ ($\{\sigma_1, \sigma_2, \sigma_3 = \sigma_2\}$) to generate symplectic $2\hbar\omega$, $4\hbar\omega$, ... excitations. Also shown are the SU(3) labels (λ, μ) and quanta distributions $\{\omega_1, \omega_2, \omega_3\}$ for some excited states.

From Eq. (2) it follows that $H(\beta_0, \beta_2)$ is not SU(3) invariant. We will now show that $H(\beta_0, \beta_2)$ exhibits partial SU(3) symmetry. Specifically, we claim that, among the eigenstates of $H(\beta_0, \beta_2)$, there exists a subset of solvable pure-SU(3) states, the $\mathrm{SU}(3) \supset \mathrm{SO}(3)$ classification of which depends on both the Elliott labels $(\lambda_\sigma, \mu_\sigma)$ of the starting state and the symplectic excitation N. In general, we find that all L states in the starting configuration ($N = 0$) are solvable with good SU(3) symmetry $(\lambda_\sigma, \mu_\sigma)$. For excited configurations ($N > 0$ and even) we distinguish between two possible cases: (a) $\lambda_\sigma > \mu_\sigma$: the pure states belong to $(\lambda, \mu) = (\lambda_\sigma - N, \mu_\sigma + N)$ and have $L = \mu_\sigma + N, \mu_\sigma + N + 1, \ldots, \lambda_\sigma - N + 1$ with $N = 2, 4, \ldots$ subject to $2N \leq (\lambda_\sigma - \mu_\sigma + 1)$. (b) $\lambda_\sigma \leq \mu_\sigma$: the special states belong to $(\lambda, \mu) = (\lambda_\sigma + N, \mu_\sigma)$ and have $L = \lambda_\sigma + N, \lambda_\sigma + N + 1, \ldots, \lambda_\sigma + N + \mu_\sigma$ with $N = 2, 4, \ldots$.

To prove the claim, it suffices to show that $\hat{B}_0$ annihilates the states in question. For $N = 0$ this follows immediately from the fact that the $0\hbar\omega$ starting configuration is a Sp(6,R) lowest weight which, by definition, is annihilated by the lowering operators of the Sp(6,R) algebra. The latter include the generators $\hat{B}_{lm}^{(02)}$. For $N > 0$, let $\{\sigma_1, \sigma_2, \sigma_3\}$ be the quanta distribution for a $0\hbar\omega$ state with $\lambda_\sigma > \mu_\sigma$. Adding N quanta to the 2-direction yields a $N\hbar\omega$ state with quanta distribution $\{\sigma_1, \sigma_2 + N, \sigma_3\}$, that is, $(\lambda, \mu) = (\lambda_\sigma - N, \mu_\sigma + N)$. Acting with the rotational invariant $\hat{B}_0$ on such a state does not affect the angular momentum, but removes two quanta from the 2-direction, giving a $(N - 2)\hbar\omega$ state with $(\lambda', \mu') = (\lambda_\sigma - N + 2, \mu_\sigma + N - 2)$. (The symplectic generator $\hat{B}_0$ cannot remove quanta from the other two directions of this particular state, since this would yield a state belonging to a different symplectic irrep.) Comparing the number of L occurrences in (λ, μ) and (λ', μ'), one finds that as long as $\lambda_\sigma - N + 1 \geq \mu_\sigma + N$, $\Delta_L(N) \equiv \kappa_L^{\max}(\lambda, \mu) - \kappa_L^{\max}(\lambda', \mu') = 1$ for $L = \mu_\sigma + N, \mu_\sigma + N + 1, \ldots, \lambda_\sigma - N + 1$, and $\Delta_L(N) = 0$ otherwise. When $\Delta_L(N) = 1$, a linear combination $|\phi_L(N)\rangle = \sum_\kappa c_\kappa |N\hbar\omega(\lambda_\sigma - N, \mu_\sigma + N)\kappa LM\rangle$ exists such that $\hat{B}_0|\phi_L(N)\rangle = 0$, and thus our claim for family (a) holds. The proof for family (b) can be carried out analogously if one considers adding N quanta to the 1-direction of the starting irrep. In this case there is no restriction on N; hence, family (b) is infinite.

The special states have well-defined symmetry $\mathrm{Sp}(6, R) \supset \mathrm{SU}(3) \supset \mathrm{SO}(3)$ and are annihilated by $\hat{B}_0$. This ensures that they are solvable eigenstates of $H(\beta_0, \beta_2)$ with eigenvalues $E(N = 0) = 0$, $E(N) = \beta_2 N(N_\sigma - \lambda_\sigma + \mu_\sigma - 6 + 3N/2)/3$ for family (a), and $E(N) = \beta_2 N(N_\sigma + 2\lambda_\sigma + \mu_\sigma - 3 + 3N/2)/3$ for family (b). All $0\hbar\omega$ states are unmixed and span the entire $(\lambda_\sigma, \mu_\sigma)$ irrep. In contrast, for the excited levels ($N > 0$), the pure states span only part of the corresponding SU(3) irreps. There are other states at each excited level which do not preserve the SU(3) symmetry and therefore contain a mixture of SU(3) irreps. The

partial SU(3) symmetry of $H(\beta_0, \beta_2)$ is converted into partial dynamical SU(3) symmetry by adding to it SO(3) rotation terms which lead to $L(L + 1)$-type splitting but do not affect the wave functions. The solvable states then form rotational bands and, since their wave functions are known, one can evaluate the $E2$ rates between them [10]. It is of interest to note that both the fermion Hamiltonian presented here and the boson Hamiltonian of [3] exhibit partial SU(3) symmetry and involve a SU(3) tensor of the form $[(2,0) \times (0,2)](2,2)L = 0$.

To illustrate that the PDS Hamiltonians of Eq. (2) are physically relevant, we compare the eigenstates of $H_{\text{PDS}} = h(N) + \xi H(\beta_0 = 12, \beta_2 = 18) + \gamma_2 \hat{L}^2 + \gamma_4 \hat{L}^4$ to those of the symplectic Hamiltonian $H_{\text{Sp6}} = \hat{H}_0 - \chi Q_2 \cdot Q_2 + d_2 \hat{L}^2 + d_4 \hat{L}^4$. Here the function $h(N)$ is simply a constant for a given $N\hbar\omega$ excitation and contains the h.o. term $\hat{H}_0$. Least squares fits to measured energies and $B(E2)$ values of the ground band of ^{20}Ne were carried out for $2\hbar\omega$, $4\hbar\omega$, $6\hbar\omega$, and $8\hbar\omega$ symplectic model spaces. The resulting energies and transition rates converge to values which agree with the data (Fig. 2 and Table I). The parameters γ_2 and γ_4 in H_{PDS} were determined by the energy splitting between states of the ground band, ξ was adjusted to reproduce the relative positions of the resonance bandheads, and $h(N)$ was fixed by the energy difference $[E(0_2^+) - E(0_1^+)]$. Figure 2 and Table I demonstrate the level of agreement between the PDS and symplectic results.

An analysis of the structure of the ground and resonance bands reveals the amount of mixing in the $8\hbar\omega$ symplectic ($Q_2 \cdot Q_2$) wave functions. Figure 3 shows the decomposition for representative (2^+) states of the five lowest rotational bands. Ground band ($K = 0_1$) states are found to have a strong $0\hbar\omega$ component ($\geq 64\%$), and three of the four resonance bands are clearly dominated ($\geq 60\%$) by $2\hbar\omega$ configurations. States of the first resonance band ($K = 0_2$), however, contain significant contributions from all but the highest $N\hbar\omega$ excitations. The
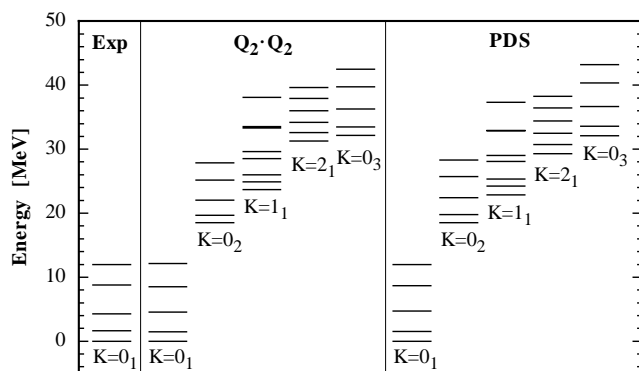
relative strengths of the SU(3) irreps within the $2\hbar\omega$ space are shown as well: states are found to be dominated by one representation [(10,0) for the $K = 0_2$ band, $(8, 1)$ for $K = 1_1$, $(6, 2)\kappa = 2$ for $K = 2_1$, and $(6, 2)\kappa = 1$ for $K = 0_3$, where $\kappa = 1$ and 2 correspond here to Vergados basis labels 0 and 2, respectively [11]], while the other irreps contribute only a few percent. Such trends are present also in the more realistic symplectic calculations of [12].

The PDS Hamiltonian H_{PDS} acts only within one oscillator shell; hence, its eigenfunctions do not contain admixtures from different $N\hbar\omega$ configurations. As expected, H_{PDS} has families of pure SU(3) eigenstates which can be organized into rotational bands. The ground band belongs entirely to $N = 0$, $(\lambda, \mu) = (8, 0)$, and all states of the $K = 2_1$ band have quantum labels $N = 2$, $(\lambda, \mu) = (6, 2)$, $\kappa = 2$. A comparison with the symplectic case shows that the $N\hbar\omega$ level to which a particular PDS band belongs is also dominant in the corresponding symplectic band. In addition, within this dominant excitation, eigenstates of H_{PDS} and H_{Sp6} have similar SU(3) distributions; in particular, both Hamiltonians favor the same $(\lambda, \mu)\kappa$ values. Significant differences in the structure of the wave functions appear, however, for the $K = 0_2$ resonance band. In the $8\hbar\omega$ symplectic calculation, this band contains almost equal contributions from the $0\hbar\omega$, $2\hbar\omega$, and $4\hbar\omega$ levels, with additional admixtures of $6\hbar\omega$ and $8\hbar\omega$ configurations, while in the PDS calculation it belongs entirely to the $2\hbar\omega$ level. These structural differences are also evident in the interband transition rates, e.g., $B(E2; K = 0_1, 2^+ \rightarrow K = 0_2, 0^+) = 2.93$ (5.69) W.u. and $B(E2; K = 0_2, 2^+ \rightarrow K = 0_1, 0^+) = 5.84$ (12.6) W.u. in the $8\hbar\omega$ (PDS) calculation, and reflect the action of the intershell coupling terms in Eq. (1). Increasing the strength χ of $Q_2 \cdot Q_2$ in H_{Sp6} will also spread the other resonance bands over many $N\hbar\omega$ excitations. The $K = 2_1$ band (which is pure in the PDS scheme) is found to resist this spreading more strongly than the other resonances. For physically relevant values of χ, the low-lying bands have the structure shown in Fig. 3.

In summary, we have introduced a family of *fermionic* Hamiltonians with partial SU(3) symmetry. Using the

FIG. 2. Energy spectra for ^{20}Ne. Comparison between experimental values (left), results from a symplectic $8\hbar\omega$ calculation (center), and a PDS calculation (right). The angular momenta of the positive parity states in the rotational bands are $L = 0, 2, 4, \ldots$ for $K = 0$ and $L = K, K + 1, K + 2, \ldots$ otherwise.

TABLE I. $B(E2)$ values (in Weisskopf units) for ground band transitions in ^{20}Ne. Compared are several symplectic calculations, PDS results, and experimental data [13]. The static quadrupole moment of the 2_1^+ state is given in the last row. PDS results are rescaled by an effective charge $e^* = 1.95$ and the symplectic calculations employ bare charges.

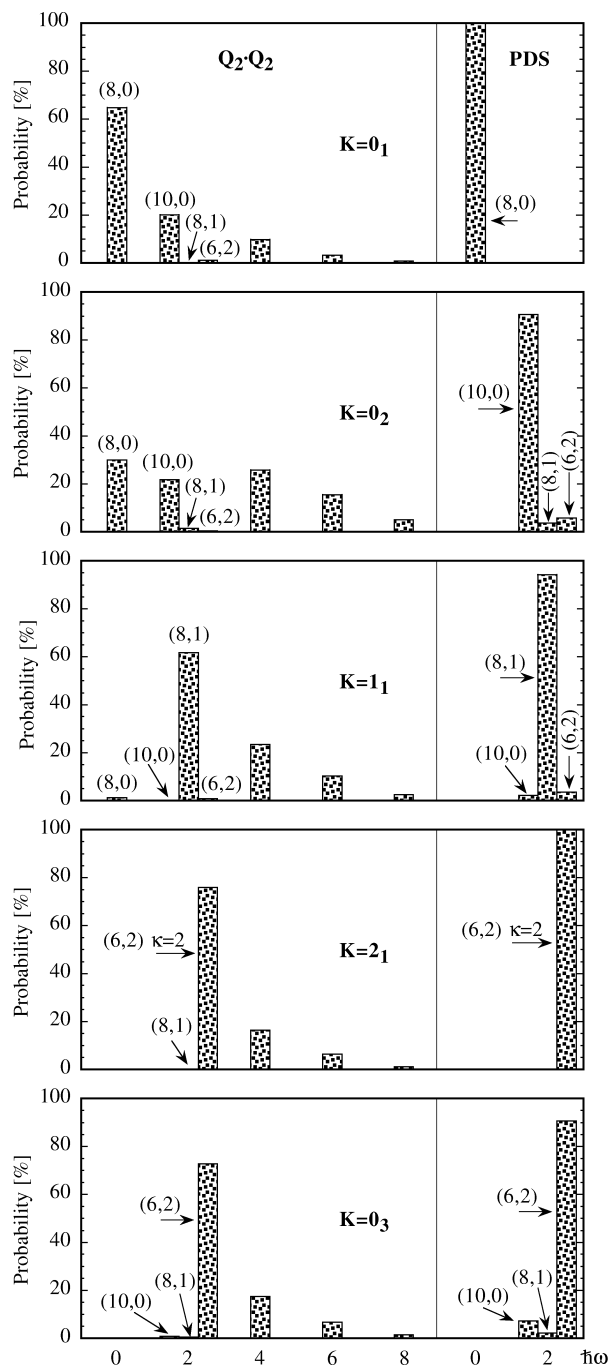| Transition	Model $B(E2)$ [W.u.]					$B(E2)$ [W.u.]
$J_i \rightarrow J_f$	$2\hbar\omega$	$4\hbar\omega$	$6\hbar\omega$	$8\hbar\omega$	PDS	Expt.
$2 \rightarrow 0$	14.0	18.7	19.1	19.3	20.3	20.3 ± 1.0
$4 \rightarrow 2$	18.4	24.5	24.6	24.5	25.7	22.0 ± 2.0
$6 \rightarrow 4$	17.1	22.3	21.5	20.9	21.8	20.0 ± 3.0
$8 \rightarrow 6$	12.4	15.2	13.3	12.4	12.9	9.0 ± 1.3
Q [eb]	-0.14	-0.16	-0.16	-0.16	-0.17	-0.23 ± 0.03

FIG. 3. Decomposition for calculated 2^+ states of ^{20}Ne. Individual contributions from the relevant SU(3) irreps at the $0\hbar\omega$ and $2\hbar\omega$ levels are shown for both a symplectic $8\hbar\omega$ calculation (denoted $Q_2 \cdot Q_2$) and a PDS calculation. In addition, the total strengths contributed by the $N\hbar\omega$ excitations for $N > 2$ are given for the symplectic case.

framework of the symplectic shell model, we have proven that these Hamiltonians possess both mixed symmetry and solvable pure-SU(3) rotational bands. For the deformed light nucleus ^{20}Ne, we have shown that various features of the quadrupole-quadrupole interaction can be reproduced with a particular parametrization of the PDS Hamiltonians.

For both the ground and the resonance bands, PDS eigenstates were seen to approximately reproduce the structure of the exact $Q_2 \cdot Q_2$ eigenstates within the $0\hbar\omega$ and $2\hbar\omega$ spaces, respectively. In particular, for each pure state of the PDS scheme we found a corresponding eigenstate of the quadrupole-quadrupole interaction, which was dominated by the same SU(3) irrep. Moreover, for reasonable interaction parameters, each rotational band was primarily located in one level of excitation, with the exception of the lowest $K = 0_2$ resonance band, which was spread over many $N\hbar\omega$ excitations. Implications of the structural differences between the various resonance bands for giant monopole and quadrupole transitions remain to be investigated. The occurrence of partial symmetries for fermions, as shown in this Letter, and for bosons, as presented in previous literature [3], highlights their relevance to dynamical systems and motivates their further study.

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