Stable Monopole and Dyon Solutions in the Einstein-Yang-Mills Theory in Asymptotically anti–de Sitter Space

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(Received 23 June 1999)

A continuum of new monopole and dyon solutions in the Einstein-Yang-Mills theory in asymptotically anti–de Sitter space are found. They are regular everywhere and specified by their mass and their non-Abelian electric and magnetic charges. A class of monopole solutions which have no node in non-Abelian magnetic fields is shown to be stable against spherically symmetric linear perturbations.

Soliton and black hole solutions to the Einstein-Yang-Mills (EYM) equations have generated considerable interest during the past decade [1–6]. In flat space there can be no static soliton solution in the pure Yang-Mills theory [7]. Even unstable static solutions cannot exist. The presence of gravity provides an attractive force which can bind non-vanishing Yang-Mills fields together. Such static solutions to the EYM equations have been found in asymptotically Minkowski or de Sitter space. They are all purely magnetic. In asymptotically Minkowski space-time the electric components are forbidden in static solutions [8]. All of these EYM solitons and black holes were shown to be unstable, where the number of unstable modes is equal to twice the number of times the magnetic component of the gauge field crosses the axis [9].

If the space-time is modified to include the cosmological constant, the no-go theorems [8] forbidding the electric components of the gauge fields fail, thus, permitting dyon solutions. We have found that in asymptotically de Sitter space-time, a nonzero electric component to the gauge fields causes $\sqrt{-g}$ to diverge at the cosmological horizon, thus excluding dyon solutions.

In this Letter, we present new monopole and dyon solutions in asymptotically anti–de Sitter (AdS) space. They are regular everywhere. In asymptotically AdS space there are solutions with no cosmological horizon and dyons solutions are allowed. Furthermore, we have found a continuum of solutions where the gauge fields have no nodes, corresponding to stable monopole and dyon solutions.

AdS space has recently attracted huge interest. The Banados-Teitelboim-Zanelli black holes in three-dimensional AdS space provide valuable information about black hole thermodynamics and quantum gravity [10]. In four dimensions, linearly stable black hole solutions have been found in asymptotically AdS space [11]. The correspondence between four-dimensional superconformal field theory and type IIB string theory on AdS_5 has been established [12]. In this Letter we show that, even in a simple Einstein-Yang-Mills system, stable monopole and dyon solutions exist in four-dimensional asymptotically AdS space, which we

believe leads to further understanding of the rich structure of quantum field theory in AdS space as well as profound implications to the evolution of the early universe.

We start with the most general expression for the spherically symmetric SU(2) gauge field in the singular gauge [13]:

$$A = \frac{1}{2e} \left[u\tau_3 dt + \nu\tau_3 dr + (w\tau_1 + \tilde{w}\tau_2)d\theta \right.$$
$$\left. + (\cot\theta\,\tau_3 + w\tau_2 - \tilde{w}\tau_1)\sin\theta\,d\phi \right], \quad (1)$$

where u, ν, w, and $\tilde{w}$ depend on r and t. The regularity at the origin imposes the boundary conditions $u = \nu = 0$ and $w^2 + \tilde{w}^2 = 1$ at $r = 0$. Under a residual U(1) gauge transformation $S = e^{i\tau_3 z(t,r)/2}$, $(u, \nu) \rightarrow (u + \dot{z}, \nu + z')$, and $w + i\tilde{w} \rightarrow e^{iz}(w + i\tilde{w})$. The spherically symmetric metric is written as

$$ds^2 = -\frac{H dt^2}{p^2} + \frac{dr^2}{H} + R^2(d\theta^2 + \sin^2\theta\,d\phi^2), \quad (2)$$

where H, p, and R are functions of t and r, in general.

We first look for static regular soliton solutions in the $\nu = 0$ gauge. Since the Yang-Mills equations imply $\tilde{w} = Cw$, where C is some constant, another gauge transformation allows us to set $C = 0$, or $\tilde{w} = 0$. Without loss of generality one can choose $w(0) = 1$. u can be interpreted as the electric part of the gauge fields and w as the magnetic part. The Schwarzshild gauge $R = r$ is taken for the metric. The coupled static EYM equations of motion are

$$\left(\frac{H}{p} w' \right)' = -\frac{p}{H} u^2 w - \frac{w}{p} \frac{(1 - w^2)}{r^2}, \quad (3)$$

$$(r^2 p u')' = \frac{2p}{H} w^2 u, \quad (4)$$

$$p' = -\frac{2\upsilon}{r} p \left[(w')^2 + \frac{u^2 w^2 p^2}{H^2} \right], \quad (5)$$

$$m' = \upsilon \left[\frac{(w^2 - 1)^2}{2r^2} + \frac{1}{2} r^2 p^2 (u')^2 \right.$$
$$\left. + H(w')^2 + \frac{u^2 w^2 p^2}{H} \right]. \quad (6)$$

Here $H(r) = 1 - 2m(r)/r - \Lambda r^2/3$ and $\upsilon = 4\pi G/e^2$. Λ is the cosmological constant. We are looking for a

solution which is regular everywhere and has finite Arnowitt-Deser-Misner (ADM) mass $m(\infty)$.

The presence of the cosmological constant affects the behavior of the soliton solutions significantly. For $\Lambda = 0$, both H and p approach a constant value as $r \to \infty$. Consequently, $w^2 \to 1$ from (3) and (6). For $\Lambda \neq 0$, w need not have an asymptotic value ± 1. It is known that Eq. (4) implies the absence of solutions with $u(r) \neq 0$ in the $\Lambda = 0$ case [8]. A similar argument applies to the $\Lambda > 0$ case in which there appears a cosmological horizon at $r = r_h$; $H(r_h) = 0$. At the horizon, either u or w must vanish to have regular solutions to Eqs. (3)–(6). Equation (4) implies that $u(r)$ is either identically zero or a monotonic function of r for $r < r_h$. Hence, $w(r_h) = 0$ and $u(r_h) \neq 0$ if $u'(0) \neq 0$. However, Eq. (3) implies

$$\frac{Hw'}{pw}\Big|_{r_1}^{r_h} = -\int_{r_1}^{r_h} dr \left\{ \frac{H}{p}\left(\frac{w'}{w}\right)^2 + \frac{pu^2}{H} + \frac{(1-w^2)}{pr^2} \right\},$$
(7)

provided that $w(r) > 0$ for $r_1 \leq r < r_h$. The right-hand side (r.h.s.) of (7) diverges, whereas the left-hand side (l.h.s.) remains finite if $w(r_h) = 0$ and $u(r_h) \neq 0$. This establishes that $u(r) = 0$ in the de Sitter space. All solutions are purely magnetic. Suppose that $0 < w^2 \leq 1$ for $0 \leq r \leq r_h$ and take $r_1 = 0$ in (7). As $w'(0) = 0$, the l.h.s. vanishes whereas the r.h.s. is negative definite. This implies that w satisfying $w^2 \leq 1$ for $r \leq r_h$ must vanish somewhere between 0 and r_h. It has been known that, in all solutions in asymptotically de Sitter space, $w(r)$ has at least one node.

The situation is quite different in asymptotically AdS space ($\Lambda < 0$). $H(r)$ is positive everywhere. All equations are consistently solved if $u(r) \sim u_0 + (u_1/r) + \ldots$ and $w(r) \sim w_0 + (w_1/r) + \ldots$ for large r. There is no restriction on the value of u_0 or w_0. Furthermore, $w(r)$ can be nodeless.

Solutions are classified by the ADM mass, $M = m(\infty)$, and electric and magnetic charges, Q_E and Q_M. From the Gauss flux theorem

$$\begin{pmatrix} Q_E \\ Q_M \end{pmatrix} = \frac{e}{4\pi} \int dS_k \sqrt{-g} \begin{pmatrix} F^{k0} \\ \tilde{F}^{k0} \end{pmatrix}$$
(8)

are conserved, but are also gauge dependent. With the ansatz in the singular gauge,

$$\begin{pmatrix} Q_E \\ Q_M \end{pmatrix} = \pm \begin{pmatrix} u_1 p_0 \\ 1 - w_0^2 \end{pmatrix} \frac{\tau_3}{2},$$
(9)

where $p(r) = p_0 + (p_1/r) + \ldots$, etc. If (u, w, m, p) is a solution, then $(-u, w, m, p)$ is also a solution. Dyon solutions come in a pair with $(\pm Q_E, Q_M, M)$.

The solutions to Eqs. (3)–(6), for $\Lambda < 0$, are evaluated numerically. The procedure is to solve these equations at $r = 0$ in terms of two free adjustable parameters a and

b and "shoot" for solutions with the desired asymptotical behavior. The behavior of solutions near the origin are

$$w(r) = 1 - br^2, \qquad m(r) = v\left(2b^2 + \frac{1}{2}a^2\right)r^3,$$

$$p(r) = 1 - (4b^2 + a^2)vr^2,$$
(10)

$$u(r) = ar + \frac{a}{5}\left[-2b + \frac{1}{3}\Lambda + 2v(a^2 + 4b^2)\right]r^3.$$

Purely magnetic solutions (monopoles) are found by setting $a = 0$ ($u = 0$). By varying the initial condition parameter b, a continuum of monopole solutions were obtained, which are similar to the black hole solutions found in Ref. [11], but are regular in the entire space. w crosses the axis an arbitrary number of times depending on the value of the adjustable shooting parameter b. Typical solutions are displayed in Fig. 1.

The behavior of m and p is similar to that of the asymptotically de Sitter solutions previously considered [5]. In contrast, as shown in Fig. 1, there exist solutions where w has no nodes. These solutions are of particular interest because they are shown to be stable against linear perturbations.

If the adjustable shooting parameter a is chosen to be nonzero, we find dyon solutions. As shown in Fig. 1, the electric component, u, of the YM fields starts at zero and monotonically increases to some finite value. The behavior of w, m, H, and p is similar to that in the monopole solutions.

Again we find a continuum of solutions where w crosses the axis an arbitrary number of times depending on a and b. Similarly there exist solutions where w does not cross the axis.
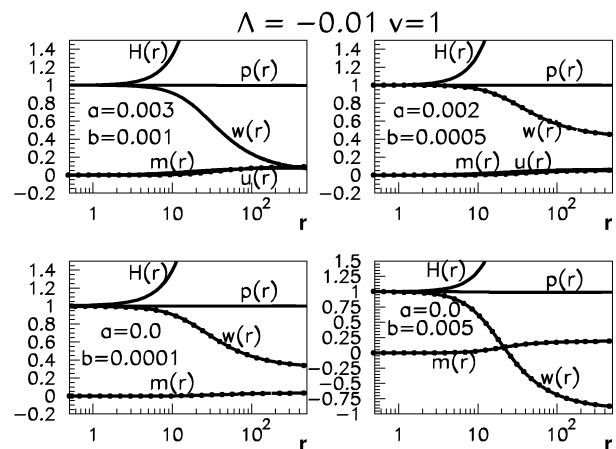

FIG. 1. Monopole and dyon solutions for $\Lambda = -0.01$ and $v = 1$. Monopoles: $(a, b) = (0, 0.001)$ and $(0, 0.005)$. (w, m) at $r = \infty$ are $(0.339, 0.034)$ and $(-0.878, 0.191)$, respectively. Dyons: $(a, b) = (0.003, 0.001)$ and $(0.002, 0.0005)$. (w, u, m) at $r = \infty$ are $(0.031, 0.080, 0.099)$ and $(0.421, 0.064, 0.056)$, respectively.
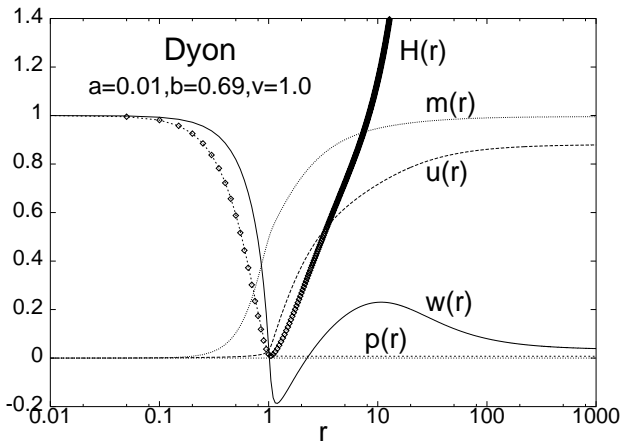
FIG. 2.   Dyon solution for $\Lambda = -0.01$, $\nu = 1$, $a = 0.01$, and $b = 0.69$. b is close to the critical value $b_c = 0.7$. H almost hits the axis around $r = 1$.

Solutions are found for a continuous set of the parameters a and b. This is in sharp contrast to the $\Lambda \geq 0$ case, in which only a discrete set of solutions is found. For some values of a and b, solutions blow up, or the function $H(r)$ crosses the axis and becomes negative. One example of solutions near the critical value $[(a, b) = (0.01, 0.69)]$ is displayed in Fig. 2. $H(r)$ becomes very close to zero at $r \sim 1$. It has $(Q_E, Q_M, M) \sim (0.015, 0.998, 0.995)$.

In Fig. 3, M is plotted as a function of Q_M for monopole solutions. The behavior of the solutions near $b = 0.7$ needs more careful analysis, although we did find that, when $b > 0.7$, all of the solutions appeared to have a horizon.

We have found that dyon solutions cover a good portion of the Q_E-Q_M plane. There are solutions with $Q_M = 0$ but $Q_E \neq 0$, which has nonvanishing $w(r)$. In the shooting parameter space (a, b), these solutions do not exactly (but almost) correspond to a universal value for $b \sim$
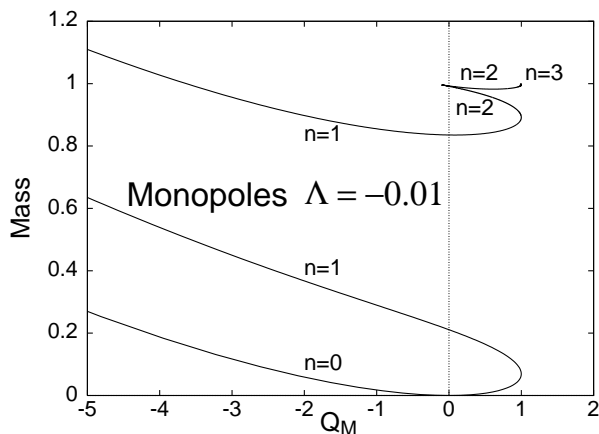
0.0054 (see Fig. 4). We do not understand why it should be so.

It has been shown that the Bartnik-McKinnon (BK) solutions and the de Sitter–EYM solutions are unstable [2,4,6,14]. In contrast, the AdS black hole solutions were shown to be stable [11] for $u = 0$. We will show that the monopole solutions without nodes presented in this paper are stable against spherically symmetric linear fluctuations.

In examining time-dependent fluctuations around monopole solutions it is convenient to work in the $u(t, r) = 0$ gauge. Solutions have nonvanishing $w(r)$, $p(r)$, and $H(r)$, but $\tilde{w}(r) = \nu(r) = 0$. Linearized equations for $\delta w(t, r)$, $\delta \tilde{w}(t, r)$, $\delta \nu(t, r)$, $\delta p(t, r)$, and $\delta H(t, r)$ have been derived in the literature [6]. Fluctuations decouple in two groups. $\delta w(t, r)$, $\delta p(t, r)$, and $\delta H(t, r)$ form even-parity perturbations, whereas $\delta \tilde{w}(t, r)$ and $\delta \nu(t, r)$ form odd-parity perturbations. The linearized equations imply that $\delta p(t, r)$, and $\delta H(t, r)$ are determined by $\delta w(t, r)$, and $\delta \tilde{w}(t, r)$ by $\delta \nu(t, r)$.

$\beta(t, r) = r^2 p \delta \nu / w = e^{-i\omega t} \beta(r)$ satisfies $\{-(d/d\rho)^2 + U_\beta(\rho)\}\beta = \omega^2 \beta$, where

$$U_\beta = \frac{H}{r^2 p^2} (1 + w^2) + \frac{2}{w^2} \left(\frac{dw}{d\rho}\right)^2 \qquad (11)$$

and $d\rho/dr = p/H$. The range of ρ is finite: $0 \leq \rho \leq \rho_{\max}$. Equation (11) is of the form of the Schrödinger equation on a one-dimensional interval. When $w(r)$ in the monopole solution has no node ($w > 0$ for all r), then $U_\beta > 0$ is a smooth potential so that β is a smooth function of ρ in the entire range satisfying $\beta\beta' = 0$ at $\rho = 0$ and $\rho_{\max}$. The eigenvalue ω^2 is positive definite, i.e., the solution is stable against odd-parity perturbations. If $w(r)$ has n nodes, i.e., $w(r_j) = 0$ ($j = 1, \ldots, n$), the potential U_β develops $(\rho - \rho_j)^{-2}$ singularities. The solution $\beta(\rho)$ to (11) is no longer regular at $\rho = \rho_j$ so that the positivity of the differential operator $-d^2/d\rho^2$ is not guaranteed. Indeed, Volkov et al. have

FIG. 3.   Mass M is plotted as a function of magnetic charge Q_M for monopole solutions. The number of nodes, n, in $w(r)$ is also marked.
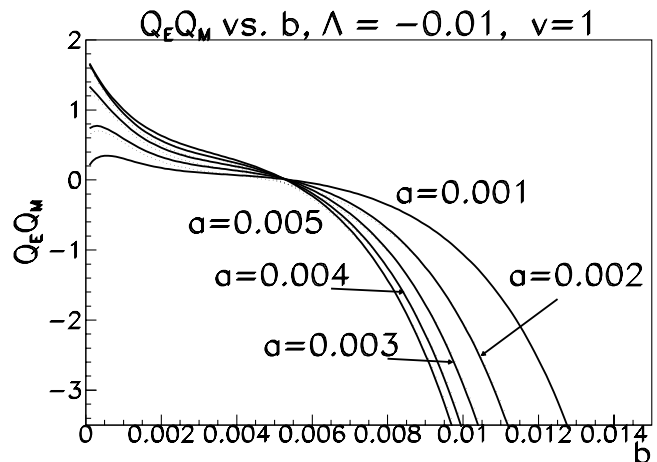

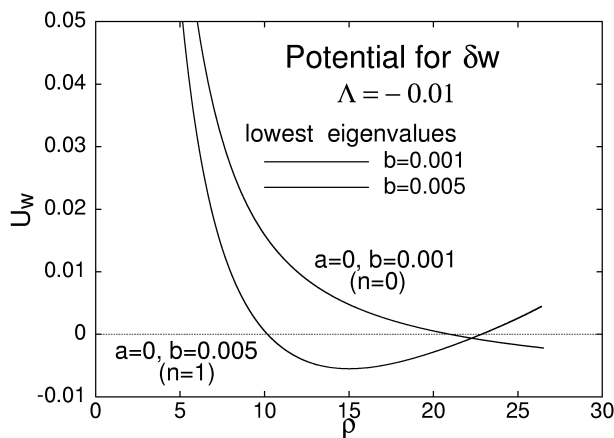FIG. 4. $Q_E \cdot Q_M$ is plotted as a function of b with a fixed. Q_M vanishes at about $b = 0.054$.

FIG. 5.   The potential $U_w(\rho)$ in (12) for the monopole solutions with $b = 0.001$ and 0.005. The number of nodes (n) in $w(r)$ is 0 and 1, respectively. The lowest eigenvalue ω^2 is found to be 0.028 or 0.023, respectively, implying the stability of the solutions.

proven for the BK solutions that there appear exactly n negative eigenmodes ($\omega^2 < 0$) if w has n nodes [9]. Their argument applies to our case without modification. One concludes that the solutions with nodes in w are unstable against parity-odd perturbations.

Similarly, parity-even perturbations $\delta w(t, r) = e^{-i\omega t}\delta w(r)$ satisfies the Schrödinger equation with a potential

$$U_w = \frac{H(3w^2 - 1)}{p^2 r^2} + 4v\frac{d}{d\rho}\left(\frac{Hw'^2}{pr}\right). \quad (12)$$

$U_w(\rho)$ is not positive definite, but is regular in the entire range $0 \leq \rho \leq \rho_{\max}$. We have solved the Schrödinger equation for δw numerically for typical monopole solutions. The potential is displayed in Fig. 5 for the solutions with $(a, b) = (0, 0.001)$ and $(0, 0.005)$. The former has no node in w, while the latter has one node. The asymptotic value $w(\infty)$ is 0.339 or -0.878, respectively. The lowest eigenvalue ω^2 is found to be 0.028 or 0.023. Hence, these solutions are stable against parity-even perturbations. Note that in the $\Lambda < 0$ case some of the $n = 1$ solutions are stable against parity-even perturbations, while in the $\Lambda \geq 0$ case they are unstable.

In this paper a continuum of new monopole and dyon solutions to the EYM equations in asymptotically AdS space have been found. There are solutions with no node in the magnetic component $w(r)$ of the SU(2) gauge fields. The

monopole solutions with no node in w have been shown to be stable against spherically symmetric perturbations. The stability of those solutions with nonzero electric fields is currently under investigation. As the monopole and dyon solutions are found in a continuum set, the dyon solutions without nodes in $w(r)$ are also expected to be stable.

The existence of stable monopole and dyon configurations may have tremendous consequences in cosmology if the early universe ever was in the AdS phase. Stable solutions exist only with a negative cosmological constant. The existence of the boundary in the AdS space must be playing a crucial role. The connection to the AdS/CFT correspondence [12] is yet to be explored. A more thorough analysis of the solutions with varying Λ as well as black hole solutions will be presented in separate publications.

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